Twisting and Rieffel's deformation of locally compact quantum groups. Deformation of the Haar measure.

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Abstract

We develop the twisting construction for locally compact quantum groups. A new feature, in contrast to the previous work of M. Enock and the second author, is a non-trivial deformation of the Haar measure. Then we construct Rieffel's deformation of locally compact quantum groups and show that it is dual to the twisting. This allows to give new interesting concrete examples of locally compact quantum groups, in particular, deformations of the classical $az + b$ group and of the Woronowicz' quantum $az + b$ group.

1 Introduction

The problem of extension of harmonic analysis on abelian locally compact (l.c.) groups, to non abelian ones, leads to the introduction of more general objects. Indeed, the set $\hat{G}$ of characters of an abelian l.c. group G is again an abelian l.c. group - the dual group of G. The Fourier transform maps functions on G to functions on $\hat{G}$, and the Pontrjagin duality theorem claims that $\hat{\hat{G}}$ is isomorphic to G. If G is not abelian, the set of its characters is too small, and one should use instead the set $\hat{G}$ of (classes of) its unitary irreducible representations and their matrix coefficients. For compact groups, this leads to the Peter-Weyl theory and to the Tannaka-Krein duality, where $\hat{G}$ is not a group, but allows to reconstruct G. Such a non-symmetric duality was established for unimodular groups by W.F. Stinespring, and for general l.c. groups by P. Eymard and T. Tatsuuma.

In order to restore the symmetry of the duality, G.I. Kac introduced in 1961 a category of *ring groups* which contained unimodular groups and their duals. The duality constructed by Kac extended those of Pontrjagin, Tannaka-Krein and Stinespring. This theory was completed in early 70-s by G.I. Kac and the second author, and independently by M. Enock and J.-M. Schwartz, in order

*Laboratoire de Mathématiques, Université de Franche-Comté, 16 route de Gray, 25030 Besancon Cedex, France. E-mail: fima@math.unicaen.fr

†Laboratoire de Mathématiques Nicolas Oresme, Université de Caen, B.P. 5186, 14032 Caen Cedex, France. E-mail: vainerman@math.unicaen.fr

to cover all l.c. groups. The objects of this category are called *Kac algebras* [2]. L.c. groups and their duals can be viewed respectively as commutative and co-commutative Kac algebras, the corresponding duality covered all known versions of duality for l.c. groups.

Quantum groups discovered by V.G. Drinfeld and others gave new important examples of Hopf algebras obtained by deformation of universal enveloping algebras and of function algebras on Lie groups. Their operator algebraic versions did not verify some of Kac algebra axioms and motivated strong efforts to construct a more general theory. Important steps in this direction were made by S.L. Woronowicz with his theory of compact quantum groups and a series of important concrete examples, S. Baaj and G. Skandalis with their fundamental concept of a multiplicative unitary and A. Van Daele who introduced an important notion of a multiplier Hopf algebra. Finally, the theory of l.c. quantum groups was proposed by J. Kustermans and S. Vaes [8], [9].

A number of "isolated" examples of non-trivial (i.e., non commutative and non cocommutative) l.c. quantum groups was constructed by S.L. Woronowicz and other people. They were formulated in terms of generators of certain Hopf $*$-algebras and commutation relations between them. It was much harder to represent them by operators acting on a Hilbert space, to associate with them an operator algebra and to construct all ingredients of a l.c. quantum group. There was no general approach to these highly nontrivial problems, and one must design specific methods in each specific case (see, for example, [19], [17]).

In [3], [16] M. Enock and the second author proposed a systematic approach to the construction of non-trivial Kac algebras by twisting. To illustrate it, consider a cocommutative Kac algebra structure on the group von Neumann algebra $M = \mathcal{L}(G)$ of a non commutative l.c. group G with comultiplication $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ (λ_g is the left translation by $g \in G$). Let us construct on M another (in general, non cocommutative) Kac algebra structure with comultiplication $\Delta_\Omega(\cdot) = \Omega\Delta(\cdot)\Omega^*$, where $\Omega \in M \otimes M$ is a unitary verifying certain 2-cocycle condition. In order to find such an Ω, let us, following to M. Rieffel [11] and M. Landstad [10], take an inclusion $\alpha : L^\infty(\hat{K}) \to M$, where $\hat{K}$ is the dual to some abelian subgroup K of G such that $\delta|_K = 1$ ($\delta(\cdot)$ is the module of G). Then, one lifts a usual 2-cocycle Ψ of $\hat{K}$: $\Omega = (\alpha \otimes \alpha)\Psi$. The main result of [3] is that Haar measure on $\mathcal{L}(G)$ gives also the Haar measure of the deformed object.

Even though a series of non-trivial Kac algebras was constructed in this way, the above mentioned "unimodularity" condition on K was restrictive. Here we develop the twisting construction for l.c. quantum groups without this condition and compute explicitly the deformed Haar measure. Thus, we are able to construct l.c. quantum groups which are not Kac algebras and to deform objects which are already non-trivial, for example, the $az + b$ quantum group [19], [17].

A dual construction that we call Rieffel's deformation of a l.c. group has been proposed in [11], [12], and [10], where, using a bicharacter on an abelian subgroup, one deforms the algebra of functions on a group. This construction has been recently developed by Kasprzak [6] who showed that the dual comultiplication is exactly the twisted comultiplication of $\mathcal{L}(G)$. Unfortunately, a trace that he constructed on the deformed algebra is invariant only under the above mentioned "unimodularity" condition. In this paper we construct Rieffel's deformation of l.c. quantum groups without this condition and compute the corresponding left invariant weight. This proves, in particular, the existence of invariant weights on the classical Rieffel's deformation. We also establish the duality between twisting and the Rieffel's deformation.

The structure of the paper is as follows. First, we recall some preliminary definitions and give our main results. In Section 3 we develop the twisting construction for l.c. quantum groups. Section 4 is devoted to the Rieffel's deformations of l.c. quantum groups and to the proof of the duality theorem. In Section 5 we present examples obtained by the two constructions: 1) from group von Neumann algebras $\mathcal{L}(G)$, in particular, when G is the $az+b$ group; 2) from the $az+b$ quantum group. Some useful technical results are collected in Appendix.

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2 Preliminaries and main results

2.1 Notations.

Let us denote by $B(H)$ the algebra of all bounded linear operators on a Hilbert space H, by $\otimes$ the tensor product of Hilbert spaces or von Neumann algebras and by Σ (resp., σ) the flip map on it. If H, K and L are Hilbert spaces and $X \in B(H \otimes L)$ (resp., $X \in B(H \otimes K), X \in B(K \otimes L)$), we denote by X_{13} (resp., $X_{12},\ X_{23}$) the operator $(1 \otimes \Sigma^*)(X \otimes 1)(1 \otimes \Sigma)$ (resp., $X \otimes 1,\ 1 \otimes X$) defined on $H \otimes K \otimes L$. The identity map will be denoted by ι.

Given a *normal semi-finite faithful* (n.s.f.) weight θ on a von Neumann algebra M, we denote: $\mathcal{M}_\theta^+ = \{x \in M^+ \mid \theta(x) < +\infty\}$, $\mathcal{N}_\theta = \{x \in M \mid x^*x \in M_\theta^+\}$, and $\mathcal{M}_\theta = span\ \mathcal{M}_\theta^+$ All l.c. groups considered in this paper are supposed to be second countable, all Hilbert spaces separable and all von Neumann algebras with separable predual.

2.2 Locally compact quantum groups [8], [9]

A pair (M, Δ) is called a (von Neumann algebraic) l.c. quantum group when

- M is a von Neumann algebra and $\Delta : M \to M \otimes M$ is a normal and unital $*$-homomorphism which is coassociative: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$
- There exist n.s.f. weights φ and ψ on M such that
  - φ is left invariant in the sense that $\varphi\big((\omega \otimes \iota)\Delta(x)\big) = \varphi(x)\omega(1)$ for all $x \in \mathcal{M}_\varphi^+$ and $\omega \in M_*^+$,

– ψ is right invariant in the sense that $\psi\big((\iota\otimes\omega)\Delta(x)\big) = \psi(x)\omega(1)$ for all $x \in \mathcal{M}_\psi^+$ and $\omega \in M_*^+$.

Left and right invariant weights are unique up to a positive scalar.

Represent M on the G.N.S. Hilbert space H of φ and define a unitary W on $H \otimes H$:

$$W^*(\Lambda(a)\otimes\Lambda(b)) = (\Lambda\otimes\Lambda)(\Delta(b)(a\otimes 1)) \quad \text{for all } a, b \in N_\varphi \, .$$

Here, Λ denotes the canonical G.N.S.-map for φ, $\Lambda\otimes\Lambda$ the similar map for $\varphi\otimes\varphi$. One proves that W satisfies the *pentagonal equation*: $W_{12}W_{13}W_{23} = W_{23}W_{12}$, and we say that W is a *multiplicative unitary.* The von Neumann algebra M and the comultiplication on it can be given in terms of W respectively as

$$M = \{(\iota\otimes\omega)(W) \mid \omega \in B(H)_*\}^{-\sigma-strong*}$$

and $\Delta(x) = W^*(1\otimes x)W$, for all $x \in M$. Next, the l.c. quantum group (M,Δ) has an antipode S, which is the unique σ-strongly* closed linear map from M to M satisfying $(\iota\otimes\omega)(W) \in \mathcal{D}(S)$ for all $\omega \in B(H)_*$ and $S(\iota\otimes\omega)(W) = (\iota\otimes\omega)(W^*)$ and such that the elements $(\iota\otimes\omega)(W)$ form a σ-strong* core for S. S has a polar decomposition $S = R\tau_{-i/2}$, where R is an anti-automorphism of M and τ_t is a one-parameter group of automorphisms of M. We call R the unitary antipode and τ_t the scaling group of (M,Δ). We have $\sigma(R\otimes R)\Delta = \Delta R$, so φR is a right invariant weight on (M,Δ), and we take $\psi := \varphi R$.

There exist a unique number $\nu > 0$ and a unique positif self-adjoint operator δ_M affiliated to M, such that $[D\psi : D\varphi]_t = \nu^{\frac{it^2}{2}}\delta_M^{it}$. ν is the scaling constant of (M,Δ) and δ_M is the modular element of (M,Δ). The scaling constant can be characterized as well by the relative invariance property $\varphi\,\tau_t = \nu^{-t}\,\varphi$.

For the dual l.c. quantum group $(\hat{M},\hat{\Delta})$ we have

$$\hat{M} = \{(\omega\otimes\iota)(W) \mid \omega \in B(H)_*\}^{-\sigma-strong*}$$

and $\hat{\Delta}(x) = \Sigma W(x\otimes 1)W^*\Sigma$ for all $x \in \hat{M}$. Turn the predual M_* into a Banach algebra with the product $\omega\,\mu = (\omega\otimes\mu)\Delta$ and define

$$\lambda : M_* \to \hat{M} : \lambda(\omega) = (\omega\otimes\iota)(W),$$

then λ is a homomorphism and $\lambda(M_*)$ is a σ-strongly* dense subalgebra of $\hat{M}$. A left invariant n.s.f. weight $\hat{\varphi}$ on $\hat{M}$ can be constructed explicitly. Let $\mathcal{I} = \{\omega \in M_* \,|\, \exists C \geq 0,\, |\omega(x^*)| \leq C||\Lambda(x)||\, \forall x \in \mathcal{N}_\varphi\}$. Then $(H,\iota,\hat{\Lambda})$ is the G.N.S. construction for $\hat{\varphi}$ where $\lambda(\mathcal{I})$ is a σ-strong-*-norm core for $\hat{\Lambda}$ and $\hat{\Lambda}(\lambda(\omega))$ is the unique vector $\xi(\omega)$ in H such that

$$\langle\xi(\omega),\Lambda(x)\rangle = \omega(x^*).$$

The multiplicative unitary of $(\hat{M},\hat{\Delta})$ is $\hat{W} = \Sigma W^*\Sigma$.

Since $(\hat{M},\hat{\Delta})$ is again a l.c. quantum group, denote its antipode by $\hat{S}$, its unitary antipode by $\hat{R}$ and its scaling group by $\hat{\tau}_t$. Then we can construct the dual of $(\hat{M}, \hat{\Delta})$, starting from the left invariant weight $\hat{\varphi}$. The bidual l.c. quantum group $(\hat{\hat{M}}, \hat{\hat{\Delta}})$ is isomorphic to (M, Δ). Denote by $\hat{\sigma}_t$ the modular automorphism group of the weight $\hat{\varphi}$. The modular conjugations of the weights φ and $\hat{\varphi}$ will be denoted by J and $\hat{J}$ respectively. Let us mention that $R(x) = \hat{J}x^*\hat{J}$, for all $x \in M$, and $\hat{R}(y) = Jy^*J$, for all $y \in \hat{M}$.

(M, Δ) is a *Kac algebra* (see [2]) if and only if $\tau_t = \iota$ and δ_M is affiliated to the center of M. In particular, (M, Δ) is a Kac algebra if M is commutative. Then (M, Δ) is generated by a usual l.c. group $G : M = L^\infty(G), (\Delta_G f)(g, h) = f(gh)$, $(S_G f)(g) = f(g^{-1})$, $\varphi_G(f) = \int f(g)\, dg$, where $f \in L^\infty(G)$, $g, h \in G$ and we integrate with respect to the left Haar measure dg on G. Then ψ_G is given by $\psi_G(f) = \int f(g^{-1})\, dg$ and δ_M by the strictly positive function $g \mapsto \delta_G(g)^{-1}$.

$L^\infty(G)$ acts on $H = L^2(G)$ by multiplication and $(W_G\xi)(g, h) = \xi(g, g^{-1}h)$, for all $\xi \in H \otimes H = L^2(G \times G)$. Then $\hat{M} = \mathcal{L}(G)$ is the group von Neumann algebra generated by the left translations $(\lambda_g)_{g\in G}$ of G and $\hat{\Delta}_G(\lambda_g) = \lambda_g \otimes \lambda_g$. Clearly, $\hat{\Delta}_G^{op} := \sigma \circ \hat{\Delta}_G = \hat{\Delta}_G$, so $\hat{\Delta}_G$ is cocommutative. Every cocommutative l.c. quantum group is obtained in this way.

2.3 q-commuting pair of operators [18]

We will use the following notion of commutation relations between unbounded operators. Let (T, S) be a pair of closed operators acting on a Hilbert space H. Suppose that $\mathrm{Ker}(T) = \mathrm{Ker}(S) = \{0\}$ and denote by $S = \mathrm{Ph}(S)|S|$ and $T = \mathrm{Ph}(T)|T|$ the polar decompositions. Let $q > 0$. We say that (T, S) is a *q-commuting pair* and we denote it by $TS = ST$, $TS^* = q^2S^*T$ if the following conditions are satisied

1. $\mathrm{Ph}(T)\mathrm{Ph}(S) = \mathrm{Ph}(S)\mathrm{Ph}(T)$ and $|T|$ and $|S|$ strongly commute.
2. $\mathrm{Ph}(T)|S|\mathrm{Ph}(T)^* = q|S|$ and $\mathrm{Ph}(S)|T|\mathrm{Ph}(S)^* = q|T|$.

If T and S are q-commuting and normal operators then the product TS is closable and its closure, always denoted by TS has the following polar decomposition $\mathrm{Ph}(TS) = \mathrm{Ph}(T)\mathrm{Ph}(S)$ and $|TS| = q^{-1}|T||S|$.

2.4 The quantum $az + b$ group [19], [17]

Let us describe an explicit example of l.c. quantum group. Let s and m be two operators defined on the canonical basis $(e_k)_{k\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$ by $se_k = e_{k+1}$ and $me_k = q^k e_k$ $(0 < q < 1)$. The G.N.S. space of the quantum $az + b$ group is $H = l^2\left(\mathbb{Z}^4\right)$, where we define the operators

$$a = m \otimes s^* \otimes 1 \otimes s \quad \text{and} \quad b = s \otimes m \otimes s \otimes 1$$

with polar decompositions $a = u|a|$ and $b = v|b|$ given by

$$\begin{aligned} |a| &= m \otimes 1 \otimes 1 \otimes 1 \quad \text{and} \quad u = 1 \otimes s^* \otimes 1 \otimes s \\ |b| &= 1 \otimes m \otimes 1 \otimes 1 \quad \text{and} \quad v = s \otimes 1 \otimes s \otimes 1. \end{aligned}$$

Then $u|b| = q|b|u,\ |a|v = qv|a|$, this is the meaning of the relations $ab = q^2ba$ and $ab^* = b^*a$. Also $\mathrm{Sp}(|a|) = \mathrm{Sp}(|b|) = \mathrm{Sp}(m) = q^{\mathbb{Z}} \cup \{0\}$, $\mathrm{Sp}(u) = \mathrm{Sp}(v) = \mathbb{S}^1$, where Sp means the spectrum. Thus, $\mathrm{Sp}(a) = \mathrm{Sp}(b) = \mathbb{C}^q \cup \{0\}$, where $\mathbb{C}^q = \left\{z \in \mathbb{C},\ |z| \in q^{\mathbb{Z}}\right\}$. The von Neumann algebra of the quantum $az + b$ group is

$$M := \left\{ \text{finite sums} \sum_{k,l} f_{k,l}(|a|, |b|)v^k u^l \quad \text{for } f_{k,l} \in L^\infty\left(q^{\mathbb{Z}} \times q^{\mathbb{Z}}\right) \right\}''.$$

Consider the following version of the quantum exponential function on $\mathbb{C}^q$:

$$F_q(z) = \prod_{k=0}^{+\infty} \frac{1 + q^{2k}\overline{z}}{1 + q^{2k}z}.$$

The fundamental unitary of the $az + b$ quantum group is $W = \Sigma V^*$ where

$$V = F_q(\hat{b} \otimes b)\chi(\hat{a} \otimes 1, 1 \otimes a),$$

and $\chi(q^{k+i\varphi}, q^{l+i\psi}) = q^{i(l\varphi+k\psi)}$ is a bicharacter on $\mathbb{C}^q$. The comultiplication is then given on generators by

$$W^*(1 \otimes a)W = a \otimes a \quad \text{and} \quad W^*(1 \otimes b)W^* = a \otimes b \dot{+} b \otimes 1,$$

where $\dot{+}$ means the closure of the sum. The left invariant weight is

$$\varphi(x) = \sum_{i,j} q^{2(j-i)} f_{0,0}(q^i, q^j), \quad \text{where} \quad x = \sum_{k,l} f_{k,l}(|a|, |b|)v^k u^l.$$

The G.N.S. construction for φ is given by (H, ι, Λ), where

$$\Lambda(x) = \sum_{k,l} q^{k+l} \xi_{k,l} \otimes e_k \otimes e_l \quad \text{with} \quad \xi_{k,l}(i,j) = q^{j-i} f_{k,l}(q^i, q^j).$$

The ingredients of the modular theory of φ are

$$J(e_r \otimes e_s \otimes e_k \otimes e_l) = e_{r-k} \otimes e_{s+l} \otimes e_{-k} \otimes e_{-l},$$
$$\nabla = 1 \otimes 1 \otimes m^{-2} \otimes m^{-2},$$

so $\sigma_t(a) = q^{-2it}a$ and $\sigma_t^{'}(b) = b$, and the modular element is $\delta = |a|^2$.

The dual von Neumann algebra is

$$\widehat{M} := \left\{ \text{finite sums} \sum_{k,l} f_{k,l}(|\hat{a}|, |\hat{b}|)\hat{v}^k \hat{u}^l \quad \text{for } f_{k,l} \in L^\infty\left(q^{\mathbb{Z}} \times q^{\mathbb{Z}}\right) \right\}''.$$

Here $\hat{a} = \hat{u}|\hat{a}|$ and $\hat{b} = \hat{v}|\hat{b}|$ are the polar decompostions of the operators

$$\hat{a} = s^* \otimes 1 \otimes 1 \otimes m, \quad \hat{b} = s^*m \otimes \left(-m^{-1} \otimes m^{-1}s^* + m^{-1}s^* \otimes s^*\right) \otimes s.$$

The formulas for the dual comultiplication and the dual left invariant weight are the same, but this time in terms of $\hat{a}$ and $\hat{b}$.

2.5 One-parameter groups of automorphisms of von Neumann algebras

Consider a von Neumann algebra $M \subset \mathcal{B}(H)$ and a continuous group homomorphism $\sigma : \mathbb{R} \to \mathrm{Aut}(M)$, $t \mapsto \sigma_t$. There is a standard way to construct, for every $z \in \mathbb{C}$, a strongly closed densely defined linear multiplicative in z operator σ_z in M. Let $\mathcal{S}(z)$ be the strip $\{y \in \mathbb{C} \mid \mathrm{Im}(y) \in [0, \mathrm{Im}(z)]\}$. Then we define :

- The domain $D(\sigma_z)$ is the set of such elements x in M that the map $t \mapsto \sigma_t(x)$ has a strongly continuous extension to $\mathcal{S}(z)$ analytic on $\mathcal{S}(z)^0$.
- Consider x in $D(\sigma_z)$ and f the unique extension of the map $t \mapsto \sigma_t(x)$ strongly continuous on $\mathcal{S}(z)$ and analytic on $\mathcal{S}(z)^0$. Then, by definition, $\sigma_z(x) = f(z)$.

If x is not in $D(\sigma_z)$, we define an unbounded operator $\sigma_z(x)$ on H as follows:

- The domain $D(\sigma_z(x))$ is the set of such $\xi \in H$ that the map $t \mapsto \sigma_t(x)\xi$ has a continuous and bounded extension to $\mathcal{S}(z)$ analytic on $\mathcal{S}(z)^0$.
- Consider ξ in $D(\sigma_z(x))$ and f the unique extension of the map $t \mapsto \sigma_t(x)\xi$ continuous and bounded on $\mathcal{S}(z)$, and analytic on $\mathcal{S}(z)^0$. Then, by definition, $\sigma_z(x)\xi = f(z)$.

Let x in M, then it is easily seen that the following element is analytic

$$x(n) := \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-nt^2} \sigma_t(x) dt.$$

The following lemma is a standard exercise:

Lemma 1 *1. $x(n) \to x$ σ-strongly-* and if $\xi \in \mathcal{D}(\sigma_z(x))$ we have $\sigma_z(x(n))\xi \to \sigma_z(x)\xi$.*

2. Let $X \subset M$ be a strongly- dense subspace of M then the set $\{x(n),\, n \in \mathbb{N},\, x \in X\}$ is a σ-strong-* core for σ_z.*

Proposition 1 *Let A be a positive self-adjoint operator affiliated with M and u a unitary in M commuting with A such that $\sigma_t(u) = uA^{it}$ for all $t \in \mathbb{R}$, then $\sigma_{-\frac{i}{2}}(u)$ is a normal operator affiliated with M and its polar decomposition is $\sigma_{-\frac{i}{2}}(u) = uA^{\frac{1}{2}}$.*

Proof. Let $\alpha \in \mathbb{R}$ and $\mathcal{D}_\alpha$ the horizontal strip bounded by $\mathbb{R}$ and $\mathbb{R} - i\alpha$. Let $\xi \in \mathcal{D}(A^{\frac{1}{2}})$. There exists a continuous bounded extension F of $t \mapsto A^{it}\xi$ on $\mathcal{D}_{\frac{1}{2}}$ analytic on $\mathcal{D}^0_{\frac{1}{2}}$ (see Lemma 2.3 in [13]). Define $G(z) = uF(z)$. Then $G(z)$ is continuous and bounded on $\mathcal{S}(-\frac{i}{2}) = \mathcal{D}_{\frac{1}{2}}$, and analytic on $\mathcal{S}(-\frac{i}{2})^0$. Moreover, $G(t) = uF(t) = uA^{it}\xi = \sigma_t(u)\xi$, so $\xi \in \mathcal{D}(\sigma_{-\frac{i}{2}}(u))$ and $\sigma_{-\frac{i}{2}}(u)\xi = G(-\frac{i}{2}) = uA^{\frac{1}{2}}\xi$. Then $uA^{\frac{1}{2}} \subset \sigma_{-\frac{i}{2}}(u)$. The other inclusion is proved in the same way. ■

2.6 The Vaes' weight

Let $M \subset \mathcal{B}(H)$ be a von Neumann algebra with a n.s.f. weight φ such that (H, ι, Λ) is the G.N.S. construction for φ. Let ∇, σ_t and J be the objects of the modular theory for φ, and δ a positive self-adjoint operator affiliated with M verifying $\sigma_t(\delta^{is}) = \lambda^{ist}\delta^{is}$, for all $s, t \in \mathbb{R}$ and some $\lambda > 0$.

Lemma 2 *[14] There exists a sequence of self-adjoint elements $e_n \in M$, analytic w.r.t. σ and commuting with any operator that commutes with δ, and such that, for all $x, z \in \mathbb{C}$, $\delta^x \sigma_z(e_n)$ is bounded with domain H, analytic w.r.t. σ and satisfying $\sigma_t(\delta^x\sigma_z(e_n)) = \delta^x\sigma_{t+z}(e_n)$, and $\sigma_z(e_n)$ is a bounded sequence which converges $*$-strongly to 1, for all $z \in \mathbb{C}$. Moreover, the function $(x, z) \mapsto \delta^x\sigma_z(e_n)$ is analytic from $\mathbb{C}^2$ to M.*

Let $N = \left\{a \in M,\ a\delta^{\frac{1}{2}} \text{ is bounded and } \overline{a\delta^{\frac{1}{2}}} \in \mathcal{N}_\varphi\right\}$. This is an ideal σ-strongly$*$ dense in M and the map $a \mapsto \Lambda(\overline{a\delta^{\frac{1}{2}}})$ is σ-strong$*$-norm closable; its closure will be denoted by Λ_δ.

Proposition 2 *[14] There exists a unique n.s.f. weight φ_δ on M such that $(H, \iota, \Lambda_\delta)$ is a G.N.S. construction for φ_δ. Moreover,*

- *the objects of the modular theory of φ_δ are $J_\delta = \lambda^{\frac{i}{4}}J$ and $\nabla_\delta = J\delta^{-1}J\delta\nabla$,*
- $[D\varphi_\delta : D\varphi]_t = \lambda^{i\frac{t^2}{2}}\delta^{it}$.

2.7 Main results

Let (M, Δ) be a l.c. quantum group with left and right invariant weights φ and $\psi = \varphi \circ R$, and the corresponding modular groups σ and σ'. Let $\Omega \in M \otimes M$ be a *2-cocycle*, i.e., a unitary such that $(\Omega \otimes 1)(\Delta \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \Delta)(\Omega)$. Then obviously $\Delta_\Omega = \Omega\Delta(.)\Omega^*$ is a comultiplication on M. If (M, Δ) is discrete quantum group and Ω is any 2-cocyle on (M, Δ), then (M, Δ_Ω) is again a discrete quantum group [1]. If (M, Δ) is not discrete, it is not known, in general, if (M, Δ_Ω) is a l.c. quantum group. Let us consider the following special construction of Ω. Let G be l.c. group and α be a unital normal faithful *-homomorphism from $L^\infty(G)$ to M such that $\alpha \otimes \alpha \circ \Delta_G = \Delta \circ \alpha$. In this case we say that G is a *co-subgroup* of (M, Δ), and we write $\widehat{G} < (M, \Delta)$. Then the von Neumann algebraic version of Proposition 5.45 in [8] gives

$$\tau_t \circ \alpha = \alpha \quad \text{and} \quad R \circ \alpha(F) = \alpha(F(\cdot^{-1})),\ \forall F \in L^\infty(G).$$

Let Ψ be a continuous bicharacter on G. Then $\Omega = (\alpha \otimes \alpha)(\Psi)$ is a 2-cocycle on (M, Δ). In [3] it was supposed that σ_t acts trivially on the image of α and it was shown that in this case, φ is also Δ_Ω-left invariant. Here we suppose that σ_t acts by translations, i.e., that there exists a continuous group homomorphism $t \mapsto \gamma_t$ from $\mathbb{R}$ to G such that $\sigma_t(\alpha(F)) = \alpha(F(.\gamma_t^{-1}))$. In this case we say that the co-subgroup G is *stable*. Then σ'_t also acts by translations:

$$\sigma'_t \circ \alpha(F) = R \circ \sigma_{-t} \circ R \circ \alpha(F) = \alpha(F(\cdot\gamma_t^{-1})) = \sigma_t \circ \alpha(F). \qquad (1)$$

In particular, $\delta^{it}\alpha(F) = \alpha(F)\delta^{it}, \forall\ t \in \mathbb{R},\ F \in L^\infty(G)$. In our case φ is not necessarily Δ_Ω-left invariant, and one has to construct another weight on M. Note that $(t,s) \mapsto \Psi(\gamma_t, \gamma_s)$ is a bicharacter on $\mathbb{R}$. Thus, there exists $\lambda > 0$ such that $\Psi(\gamma_t, \gamma_s) = \lambda^{its}$ for all $s, t \in \mathbb{R}$. Let us define the following unitaries in M:

$$u_t = \lambda^{i\frac{t^2}{2}}\alpha\left(\Psi(\cdot, \gamma_t^{-1})\right) \quad \text{and} \quad v_t = \lambda^{i\frac{t^2}{2}}\alpha\left(\Psi(\gamma_t^{-1}, \cdot)\right).$$

Then equation (1) and the definition of a bicharacter imply that u_t is a σ-cocycle and v_t is a σ'-cocycle. The converse of the Connes' Theorem gives then n.s.f. weights φ_Ω and ψ_Ω on M such that:

$$u_t = [D\varphi_\Omega : D\varphi]_t \quad \text{and} \quad v_t = [D\psi_\Omega : D\psi]_t.$$

The main result of Section 3 is the following. We denote by W the multiplicative unitary of (M, Δ), and put $W_\Omega^* = \Omega(\hat{J} \otimes J)W\tilde{\Omega}(\hat{J} \otimes J)$.

Theorem 1 *(M, Δ_Ω) is a l.c. quantum group :*

- *φ_Ω is left invariant,*
- *ψ_Ω is right invariant,*
- *W_Ω is the fundamental multiplicative unitary.*
- *The scaling group and the scaling constant are $\tau_t^\Omega = \tau_t$, $\nu_\Omega = \nu$.*

If G is abelian, we compute explicitly the modular element and the antipode.

In section 4 we construct the Rieffel's deformation of a l.c. quantum group with an abelian stable co-subgroup $\widehat{G} < (M, \Delta)$ and prove that this construction is dual to the twisting. Switching to the additive notations for G, define $L_\gamma = \alpha(u_\gamma)$ and $R_\gamma = JL_\gamma J$, where $\gamma \in \hat{G}$, $u_\gamma = \langle \gamma, g \rangle \in L^\infty(G)$, and J is the modular conjugation of φ. Then Proposition 3 shows that $\widehat{G}^2$ acts on $\widehat{M}$ by conjugation by the unitaries $L_{\gamma_1}R_{\gamma_2}$. We call this action the *left-right action.* Let $N = \widehat{G}^2 \ltimes \widehat{M}$ be the crossed product von Neumann algebra generated by $\lambda_{\gamma_1,\gamma_2}$ and $\pi(x)$, where $\gamma_i \in \widehat{G}$ and $x \in \widehat{M}$, and let θ be the dual action of G^2 on N. We show that there exists a unique unital normal *-homomorphism Γ from N to $N \otimes N$ such that $\Gamma(\lambda_{\gamma_1,\gamma_2}) = \lambda_{\gamma_1,0} \otimes \lambda_{0,\gamma_2}$ and $\Gamma(\pi(x)) = (\pi \otimes \pi)\hat{\Delta}(x)$. Let Ψ be a continuous bicharacter on G. Note that, for all $g \in G$, we have $\Psi_g \in \widehat{G}$, where $\Psi_g(h) = \Psi(h, g)$. We denote by θ^Ψ the *twisted dual action* of G^2 on N:

$$\theta^\Psi_{(g_1,g_2)}(x) = \lambda_{\Psi_{g_1},\Psi_{g_2}}\theta_{(g_1,g_2)}(x)\lambda^*_{\Psi_{g_1},\Psi_{g_2}}, \quad \text{for any } g_1, g_2 \in G,\ x \in \widehat{G}^2 \ltimes \widehat{M}, \tag{2}$$

and by N_Ω the fixed point algebra under this action (we would like to point out that N_Ω is not a deformation of N, it is just a fixed point algebra with respect to the action θ^Ψ related to Ω). Put $\Upsilon = (\lambda_R \otimes \lambda_L)(\tilde{\Psi}^*) \in N \otimes N$, where λ_R and λ_L are the unique unital normal *-homomorphisms from $L^\infty(G)$ to N such that $\lambda_L(u_\gamma) = \lambda_{\gamma,0}$ and $\lambda_R(u_\gamma) = \lambda_{0,\gamma}$, and put $\Gamma_\Omega(\cdot) = \Upsilon\Gamma(\cdot)\Upsilon^*$. Then we show that Γ_Ω is a comultiplication on N_Ω and construct a left invariant weight on $(N_\Omega, \Gamma_\Omega)$. Because $\theta^\Psi_{g_1,g_2}(\lambda_{\gamma_1,\gamma_2}) = \theta g_1, g_2(\lambda_{\gamma_1,\gamma_2}) = \overline{\langle\gamma_1, g_1\rangle\langle\gamma_2, g_2\rangle}\lambda_{\gamma_1,\gamma_2}$, we have a canonical isomorphism $N = \widehat{G}^2 \ltimes \widehat{M} \to \widehat{G}^2 \ltimes N_\Omega$ intertwining the twisted dual action on N with the dual action on $\widehat{G}^2 \ltimes N_\Omega$. Denoting by $\widetilde{\hat{\varphi}}$ the dual weight of $\hat{\varphi}$ on N and by $\widetilde{\sigma}$ its modular group, we show that $w_t = \lambda^{-it^2}\lambda_R(\Psi(-\gamma_t, .))$ is a $\widetilde{\sigma}$-cocycle. This implies the existence of a unique n.s.f. $\tilde{\mu}_\Omega$ on N such that $w_t = [D\tilde{\mu}_\Omega : D\tilde{\hat{\varphi}}]_t$. Moreover, one can show that $\tilde{\mu}_\Omega$ is θ^Ψ invariant. Thus, there exists a unique n.s.f. μ_Ω on N_Ω such that the dual weight of μ_Ω is $\tilde{\mu}_\Omega$. In order to formulate the main result of Section 4, let us denote by $(\widehat{M}_\Omega, \hat{\Delta}_\Omega)$ the dual of (M, Δ_Ω).

Theorem 2 *$(N_\Omega, \Gamma_\Omega)$ is a l.c. quantum group and μ_Ω is left invariant. Moreover there is a canonical isomorphism $(N_\Omega, \Gamma_\Omega) \simeq (\widehat{M}_\Omega, \hat{\Delta}_\Omega)$.*

Note that the Rieffel's deformation in the C^*-setting was constructed by the first author in [4], see also Remark 4 and [5] for an overview.

In Section 5 we calculate explicitly two examples. It is known that if H is an abelian closed subgroup of a l.c. group G, then there is a unique faithful unital normal *-homomorphism α from $L^\infty(\widehat{H})$ to $\mathcal{L}(G)$ such that $\alpha(u_h) = \lambda_G(h)$, for all $h \in \widehat{H}$, where λ_G is the left regular representation of G, so $H < (\mathcal{L}(G), \hat{\Delta}_G)$ is a co-subgroup. The left (and right) invariant weight on $\mathcal{L}(G)$ is the Plancherel weight for which $\sigma_t(\lambda_g) = \delta_G^{it}(g)\lambda_g$, for all $g \in G$, where δ_G is the modular function of G. Then $\sigma_t \circ \alpha(u_g) = \alpha(u_g(\cdot - \gamma_t))$, where γ_t is the character on K defined by $\langle\gamma_t, g\rangle = \delta_G^{-it}(g)$. Because the vector space spanned by the u_h for $h \in H$ is dense in $L^\infty(\widehat{H})$, we have $\sigma_t \circ \alpha(F) = \alpha(F(\cdot - \gamma_t))$, for all $F \in L^\infty(\hat{K})$. Thus, $H < (\mathcal{L}(G), \hat{\Delta}_G)$ is stable. So, given a bicharacter Ψ on $\widehat{H}$, we can perform the twisting construction. The deformation of the Haar weight will be non trivial when H is not in the kernel of the modular function of G.

Let $G = \mathbb{C}^* \ltimes \mathbb{C}$ be the $az + b$ group and $H = \mathbb{C}^*$ be the abelian closed subgroup of elements of the form $(z, 0)$ with $z \in \mathbb{C}^*$. Identifying $\widehat{\mathbb{C}^*}$ with $\mathbb{Z} \times \mathbb{R}^*_+$:

$$\mathbb{Z} \times \mathbb{R}^*_+ \to \widehat{\mathbb{C}^*}, \quad (n, \rho) \mapsto \gamma_{n,\rho} = (re^{i\theta} \mapsto e^{i \ln r \ln \rho} e^{in\theta}),$$

let us define, for all $x \in \mathbb{R}$, the following bicharacter on $\mathbb{Z} \times \mathbb{R}^*_+$:

$$\Psi_x((n, \rho), (k, r)) = e^{ix(k \ln \rho - n \ln r)}$$

and perform the twisting construction. We obtain a family of l.c. quantum groups (M_x, Δ_x) with trivial scaling group and scaling constant. Moreover, we show that the antipode is not deformed. The main result of Section 5.1 is the following. Let us denote by φ the Plancherel weight on $\mathcal{L}(G)$ and by a subscript x the objects associated with (M_x, Δ_x).

Theorem 3 *We have:*

- $[D\varphi_x : D\varphi]_t = \lambda^G_{(e^{itx},0)}$, $\delta_x^{it} = \lambda^G_{(e^{-2itx},0)}$.

- *$(M_{-x},\Delta_{-x}) \simeq (M_x,\Delta_x)^{op}$ and if $x,y \geq 0$ with $x \neq y$ then (M_x,Δ_x) and (M_y,Δ_y) are not isomorphic.*

The von Neumann algebra of the dual quantum group $(\widehat{M_x},\hat{\Delta}_x)$ is generated by two operators $\hat{\alpha}$ and $\hat{\beta}$ affiliated with it and such that

- *$\hat{\alpha}$ is normal, $\hat{\beta}$ is q-normal, i.e., $\hat{\beta}\hat{\beta}^* = q\hat{\beta}^*\hat{\beta}$,*
- *$\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha}$ and $\hat{\alpha}\hat{\beta}^* = q\hat{\beta}^*\hat{\alpha}$, with $q = e^{4x}$.*

The comultiplication is given by $\hat{\Delta}_x(\hat{\alpha}) = \hat{\alpha}\otimes\hat{\alpha}$ and $\hat{\Delta}_x(\hat{\beta}) = \hat{\alpha}\otimes\hat{\beta}\dot{+}\hat{\beta}\otimes 1$.

For the dual $(\widehat{M_x},\hat{\Delta}_x)$ we deform, like in the Woronowicz' quantum $az+b$ group, the commutativity relation between the two coordinate functions, but the difference is that we also deform the normality of the second coordinate function.

The second example of Section 5 is the twisting of an already non trivial object. Consider the Woronowicz' quantum $az+b$ group (M,Δ) at a fixed parameter $0 < q < 1$. Let $\alpha : L^\infty(\mathbb{C}^q) \to M$ be the normal faithful *-homomorphism defined by $\alpha(F) = F(a)$. Because $\Delta(a) = a\otimes a$, one has $\Delta\circ\alpha = (\alpha\otimes\alpha)\circ\Delta_{\mathbb{C}^q}$. Thus, we have a co-subgroup $\widehat{\mathbb{C}^q} < (M,\Delta)$ which is stable:

$$\sigma_t\circ\alpha(F) = \sigma_t(F(a)) = F(\sigma_t(a)) = F(q^{-2it}a) = \alpha(F(\cdot\gamma_t^{-1})),$$

where $\gamma_t = q^{2it}\in\mathbb{C}^q$. Performing the twisting construction with the bicharacters

$$\Psi_x(q^{k+i\varphi},q^{l+i\psi}) = q^{ix(k\psi - l\varphi)},\ \forall x\in\mathbb{Z},$$

we get the twisted l.c. quantum groups (M_x,Δ_x). The main result of Section 5.2 is the following. Recall that we denote by $a = u|a|$ the polar decomposition of a.

Theorem 4 *One has $\Delta_x(a) = a\otimes a$ and $\Delta_x(b) = u^{-x+1}|a|^{x+1}\otimes b\dot{+}b\otimes u^x|a|^{-x}$. The modular element $\delta_x = |a|^{4x+2}$, the antipode is not deformed and we have $[D\varphi_x : D\varphi]_t = |a|^{-2ixt}$. Moreover, for any $x,y\in\mathbb{N}$, one has: if $x\neq y$, then (M_x,Δ_x) and (M_y,Δ_y) are not isomorphic; if $x\neq 0$, then (M_x,Δ_x) and (M_{-x},Δ_{-x}) are not isomorphic. The von Neumann algebra of the dual quantum group $(\widehat{M_x},\hat{\Delta}_x)$ is generated by two operators $\hat{\alpha}$ and $\hat{\beta}$ affiliated with it and such that*

- *$\hat{\alpha}$ is normal, $\hat{\beta}$ is p-normal, i.e., $\hat{\beta}\hat{\beta}^* = p\hat{\beta}^*\hat{\beta}$,*
- *$\hat{\alpha}\hat{\beta} = q^2\hat{\beta}\hat{\alpha}$ and $\hat{\alpha}\hat{\beta}^* = p\hat{\beta}^*\hat{\alpha}$, with $p = q^{-4x}$.*

The comultiplication is given by $\hat{\Delta}_x(\hat{\alpha}) = \hat{\alpha}\otimes\hat{\alpha}$ and $\hat{\Delta}_x(\hat{\beta}) = \hat{\alpha}\otimes\hat{\beta}\dot{+}\hat{\beta}\otimes 1$.

We refer to [5] for the explicit example of the twisting in the C^*-setting of the group G of 2×2 upper triangular matrices of determinant 1 with the abelian subgroup of diagonal matrices in G. Next subsection contains useful technical result.

2.8 Abelian stable co-subgroups

Let $\widehat{G} < (M, \Delta)$ be a stable co-subgroup with G abelian. For all $\gamma \in \hat{G}$, the map $t \mapsto \langle \gamma, \gamma_t \rangle$ is a character on $\mathbb{R}$, so there exists $\lambda(\gamma) > 0$ such that $\langle \gamma, \gamma_t \rangle = \lambda(\gamma)^{it}$ for all $t \in \mathbb{R}$.

Proposition 3 *Let $\widehat{G} < (M, \Delta)$ be a co-subgroup with G abelian. Then:*

$$1. (1 \otimes L_\gamma)W(1 \otimes L_\gamma^*) = W(L_\gamma \otimes 1), \quad (1 \otimes R_\gamma)W(1 \otimes R_\gamma^*) = (L_{-\gamma} \otimes 1)W, \tag{3}$$

for all $\gamma \in \widehat{G}$, so we have two commuting actions α^L and α^R of $\hat{G}$ on $\hat{M}$: $\alpha_\gamma^L(x) = L_\gamma x L_\gamma^*$ and $\alpha_\gamma^R(x) = R_\gamma x R_\gamma^*$. This gives an action of $\hat{G}^2$ on $\hat{M}$ $\alpha_{\gamma_1,\gamma_2} = \alpha_{\gamma_1}^L \circ \alpha_{\gamma_2}^R$ such that

$$(\iota \otimes \alpha_{\gamma_1,\gamma_2})(W) = (L_{\gamma_2}^* \otimes 1)W(L_{\gamma_1}^* \otimes 1). \tag{4}$$

2. If $\widehat{G} < (M, \Delta)$ is stable, then, for all $x \in \mathcal{N}_{\hat{\varphi}}$ and all $\gamma \in \hat{G}$, we have $\alpha_\gamma^L(x), \alpha_\gamma^R(x) \in \mathcal{N}_{\hat{\varphi}}$, $L_\gamma \hat{\Lambda}(x) = \hat{\Lambda}(\alpha_\gamma^L(x))$, and $R_\gamma \hat{\Lambda}(x) = \lambda(\gamma)^{-\frac{1}{2}} \hat{\Lambda}(\alpha_\gamma^R(x))$.

Proof. Since $\Delta(L_\gamma) = L_\gamma \otimes L_\gamma$, $\Delta(x) = W^*(1 \otimes x)W$ and $(\hat{J} \otimes J)W(\hat{J} \otimes J) = W^*$, it is easy to check the first two equalities. The equality for α follows immediately. To prove the second assertion we need the following

Lemma 3 ([8]) *Let $\omega \in \mathcal{I}$, $a \in M$, and $b \in \mathcal{D}(\sigma_{-\frac{i}{2}})$, then $a\omega b \in \mathcal{I}$ and*

$$\xi(a\omega b) = aJ\sigma_{-\frac{i}{2}}(b)^* J\xi(\omega).$$

Let us prove the second assertion. By the first assertion we have $\alpha_\gamma^L((\omega \otimes \iota)(W)) = (L_\gamma \omega \otimes \iota)(W)$. Take $\omega \in \mathcal{I}$. By Lemma 3, we have $L_\gamma \omega \in \mathcal{I}$ and

$$\hat{\Lambda}(\alpha_\gamma^L(\lambda(\omega))) = \hat{\Lambda}(\lambda(L_\gamma \omega)) = L_\gamma \hat{\Lambda}(\lambda(\omega)).$$

Because $\lambda(\mathcal{I})$ is a core for $\hat{\Lambda}$, for all $x \in \mathcal{N}_{\hat{\varphi}}$, we have $\alpha_\gamma^L(x) \in \mathcal{N}_{\hat{\varphi}}$ and

$$\hat{\Lambda}(\alpha_\gamma^L(x)) = L_\gamma \hat{\Lambda}(x).$$

By the first assertion, we have $\alpha_\gamma^R((\omega \otimes \iota)(W)) = (\omega L_{-\gamma} \otimes \iota)(W)$. Note that $\sigma_t(L_\gamma) = \lambda(\gamma)^{-it} L_\gamma$, thus $L_\gamma \in \mathcal{D}(\sigma_{\frac{i}{2}})$ and $\sigma_{\frac{i}{2}}(L_\gamma) = \lambda(\gamma)^{\frac{1}{2}} L_\gamma$. Take $\omega \in \mathcal{I}$. By Lemma 3, we have $\omega L_{-\gamma} \in \mathcal{I}$ and

$$\hat{\Lambda}(\alpha_\gamma^R(\lambda(\omega))) = \hat{\Lambda}(\lambda(\omega L_{-\gamma})) = \lambda(\gamma)^{\frac{1}{2}} R_\gamma \hat{\Lambda}(\lambda(\omega)).$$

Because $\lambda(\mathcal{I})$ is a core for $\hat{\Lambda}$, this concludes the proof. ■

3 Twisting of locally compact quantum groups

Let G be a l.c. group and (M, Δ) a l.c. quantum group. Suppose that $\widehat{G} < (M, \Delta)$ is a stable co-subgroup. We keep the notations of Section 2.7. Note that the maps $(t \mapsto \alpha(\Psi(\cdot, \gamma_t^{-1})))$ and $(t \mapsto \alpha(\Psi(\gamma_s^{-1}, \cdot)))$ are unitary representations of $\mathbb{R}$ in M. Let A and B be the positive self-adjoint operators affiliated with M such that $A^{it} = \alpha(\Psi(\cdot, \gamma_t^{-1}))$ and $B^{is} = \alpha(\Psi(\gamma_s^{-1}, \cdot))$. We have $\Delta(A) = A \otimes A$, $\Delta(B) = B \otimes B$. Note that $\sigma_t(A^{is}) = \alpha(\Psi(\cdot\gamma_t^{-1}, \gamma_s^{-1})) = \lambda^{ist} A^{is}$. Also we have $\sigma_s^{'}(B^{it}) = \sigma_s(B^{it}) = \lambda^{ist} B^{it}$, so the weights φ_Ω and ψ_Ω are the Vaes' weights associated with φ, λ and A, and with ψ, λ and B, respectively. In the sequel, we denote by Λ_Ω the canonical G.N.S. map associated with φ_Ω, and by $F \mapsto \tilde{F}$ the *-automorphism of $L^\infty(G \times G)$ defined by $\tilde{F}(g, h) = F(g^{-1}, gh)$. Theorem 1 is in fact a corollary of the following result.

Theorem 5 *For all $x, y \in \mathcal{N}_{\varphi_\Omega}$, we have $\Delta_\Omega(x)(y \otimes 1) \in \mathcal{N}_{\varphi_\Omega \otimes \varphi_\Omega}$ and*

$$(\Lambda_\Omega \otimes \Lambda_\Omega)\left(\Delta_\Omega(x)(y \otimes 1)\right) = W_\Omega^*(\Lambda_\Omega(y) \otimes \Lambda_\Omega(x)),$$

where $W_\Omega^* = \Omega(\hat{J} \otimes J) W \tilde{\Omega} (\hat{J} \otimes J)$.

Proof. Let us introduce the sets

$$N = \left\{ x \in M,\, xA^{\frac{1}{2}} \text{ is bounded and } \overline{xA^{\frac{1}{2}}} \in \mathcal{N}_\varphi \right\} \quad \text{and}$$

$$L = \left\{ x \in N,\, A^{-\frac{1}{2}} \overline{xA^{\frac{1}{2}}} \text{ is bounded and } \Lambda(\overline{xA^{\frac{1}{2}}}) \in \mathcal{D}(A^{-\frac{1}{2}}) \right\}.$$

When $y \in L$, we denote the closure of $A^{-\frac{1}{2}} \overline{xA^{\frac{1}{2}}}$ by $\overline{A^{-\frac{1}{2}} y A^{\frac{1}{2}}}$. By definition, N is a σ-strong$*$-norm core for Λ_Ω, and Proposition 18 shows that the same is true for L. As Λ_Ω is closed in these topologies, it suffices to prove the theorem for elements $x \in N$ and $y \in L$. The first step is as follows.

Lemma 4 *Let $x \in N$, $y \in L$ and $F \in (\alpha \otimes \alpha)(L^\infty(G \times G))$. Then*

$$(\Delta(x) F^* y \otimes 1)(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}}) \quad \textit{is bounded and}$$

$$\overline{(\Delta(x) F^* (y \otimes 1))(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}})} = \Delta(\overline{xA^{\frac{1}{2}}}) F^* \left(\overline{A^{-\frac{1}{2}} y A^{\frac{1}{2}}} \otimes 1 \right).$$

Proof. Note that $\Delta(A^{\frac{1}{2}}) = A^{\frac{1}{2}} \otimes A^{\frac{1}{2}} = W^* \left(1 \otimes A^{\frac{1}{2}}\right) W$. Let $x \in N$ and $\xi \in \mathcal{D}(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}})$. Then $W\xi \in \mathcal{D}(1 \otimes A^{\frac{1}{2}})$ and

$$\begin{aligned} \Delta(x)(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}})\xi &= W^*(1 \otimes x) W W^* (1 \otimes A^{\frac{1}{2}}) W \xi \\ &= W^*(1 \otimes x)(1 \otimes A^{\frac{1}{2}}) W \xi \\ &= W^*(1 \otimes \overline{xA^{\frac{1}{2}}}) W \xi = \Delta(\overline{xA^{\frac{1}{2}}})\xi. \end{aligned}$$

Thus, $\Delta(x)(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}}) \subset \Delta(\overline{xA^{\frac{1}{2}}})$ and because it is densely defined, we have shown that, $\forall x \in N$, $\Delta(x)(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}})$ is bounded and $\overline{\Delta(x)(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}})} = \Delta(\overline{xA^{\frac{1}{2}}})$. If $x \in N,\ y \in N'$, the commutativity of $(\alpha \otimes \alpha)(L^\infty(G \times G))$ implies:

$$\begin{aligned}(\Delta(x)F^*(y \otimes 1))(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}}) &= \Delta(x)(1 \otimes A^{\frac{1}{2}})F^*(yA^{\frac{1}{2}} \otimes 1) \\ &= \Delta(x)(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}})(A^{-\frac{1}{2}} \otimes 1)F^*(yA^{\frac{1}{2}} \otimes 1) \\ &= \Delta(x)(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}})F^*(A^{-\frac{1}{2}}yA^{\frac{1}{2}} \otimes 1) \\ &\subset \Delta(\overline{xA^{\frac{1}{2}}})F^*\left(\overline{A^{-\frac{1}{2}}yA^{\frac{1}{2}}} \otimes 1\right).\end{aligned}$$

Since $(\Delta(x)F^*(y \otimes 1))(A^{\frac{1}{2}} \otimes A^{\frac{1}{2}})$ is densely defined, the proof is finished. ■ In what follows, we identify $L^\infty(G)$ with its image $\alpha(L^\infty(G))$. Note that

$$(\iota \otimes \sigma_t)(\tilde{F}) = \widetilde{(\iota \otimes \sigma_t)}(F), \quad \text{for all } \ t \in \mathbb{R}, F \in L^\infty(G \times G).$$

By analytic continuation, this is also true for $t = z \in \mathbb{C}$ and $F \in \mathcal{D}(\iota \otimes \sigma_z)$.

Now we construct a set of certain elements of $\mathcal{N}_{\varphi_\Omega \otimes \varphi_\Omega}$ and give their images by $\Lambda_\Omega \otimes \Lambda_\Omega$.

Lemma 5 *Let $x \in N$, $y \in L$ and $F \in L^\infty(G \times G)$. If $F \in \mathcal{D}(\iota \otimes \sigma_{-\frac{i}{2}})$ then*

$$\Delta(x)F^*(y \otimes 1) \in \mathcal{N}_{\varphi_\Omega \otimes \varphi_\Omega} \quad \textit{and}$$

$$(\Lambda_\Omega \otimes \Lambda_\Omega)(\Delta(x)F^*(y \otimes 1)) = (\hat{J} \otimes J)W(\iota \otimes \sigma_{-\frac{i}{2}})(\tilde{F})(\hat{J} \otimes J)\left(A^{-\frac{1}{2}}\Lambda_\Omega(y) \otimes \Lambda_\Omega(x)\right).$$

Proof. According to Proposition 20 and Lemma 4, it suffices to show that

$$\begin{aligned}\forall F \in \mathcal{D}(\iota \otimes \sigma_{-\frac{i}{2}}),\ \Delta(\overline{xA^{\frac{1}{2}}})F^*(\overline{A^{-\frac{1}{2}}yA^{\frac{1}{2}}} \otimes 1) \in \mathcal{N}_{\varphi \otimes \varphi} \quad \text{and,} \\ (\Lambda \otimes \Lambda)\left(\Delta(\overline{xA^{\frac{1}{2}}})F^*(\overline{A^{-\frac{1}{2}}yA^{\frac{1}{2}}} \otimes 1)\right) \qquad (5) \\ = (\hat{J} \otimes J)W(\iota \otimes \sigma_{-\frac{i}{2}})(\tilde{F})(\hat{J} \otimes J)\left(A^{-\frac{1}{2}}\Lambda_\Omega(y) \otimes \Lambda_\Omega(x)\right).\end{aligned}$$

Let $F \in L^\infty(G \times G)$. We identify σ with its restriction to $L^\infty(G)$. A direct application of Lemma 1 (2) gives that $L^\infty(G) \odot \mathcal{D}(\sigma_{-\frac{i}{2}})$ is a σ-strong* core for $\iota \otimes \sigma_{-\frac{i}{2}}$. Taking into account the observation preceding this lemma and because $\Lambda \otimes \Lambda$ is σ-strong*-norm closed, it suffices to show (5) for $F \in L^\infty(G) \odot \mathcal{D}(\sigma_{-\frac{i}{2}})$. By linearity, we only have to show (5) for F of the form $F = F_1 \otimes F_2$ with $F_1, F_2 \in L^\infty(G)$ and $F_2 \in \mathcal{D}(\sigma_{-\frac{i}{2}})$. Proposition 18 gives $\overline{A^{-\frac{1}{2}}yA^{\frac{1}{2}}} \in \mathcal{N}_\varphi$, so $\Delta(\overline{xA^{\frac{1}{2}}})(F_1^*\overline{A^{-\frac{1}{2}}yA^{\frac{1}{2}}} \otimes 1) \in \mathcal{N}_{\varphi \otimes \varphi}$, and writing

$$\Delta(\overline{xA^{\frac{1}{2}}})(F_1^* \otimes F_2^*)(\overline{A^{-\frac{1}{2}}yA^{\frac{1}{2}}} \otimes 1) = \Delta(\overline{xA^{\frac{1}{2}}})(F_1^*\overline{A^{-\frac{1}{2}}yA^{\frac{1}{2}}} \otimes 1)(1 \otimes F_2^*)$$

with $1\otimes F_2 \in \mathcal{D}(\iota\otimes\sigma_{-\frac{i}{2}})$, we see that $\Delta(\overline{xA^{\frac12}})F^*(\overline{A^{-\frac12}yA^{\frac12}}\otimes 1) \in \mathcal{N}_{\varphi\otimes\varphi}$ and

$$
\begin{aligned}
&(\Lambda\otimes\Lambda)\left(\Delta(\overline{xA^{\frac12}})F^*(\overline{A^{-\frac12}yA^{\frac12}}\otimes 1)\right)\\
&= \left(1\otimes J\sigma_{-\frac{i}{2}}(F_2)J\right)(\Lambda\otimes\Lambda)\left(\Delta(\overline{xA^{\frac12}})(F_1^*\overline{A^{-\frac12}yA^{\frac12}}\otimes 1)\right)\\
&= \left(1\otimes J\sigma_{-\frac{i}{2}}(F_2)J\right)W^*\Lambda(F_{(1)}^*\overline{A^{-\frac12}yA^{\frac12}})\otimes\Lambda(\overline{xA^{\frac12}})\\
&\quad \text{(by definition of } W)\\
&= \left(1\otimes J\sigma_{-\frac{i}{2}}(F_2)J\right)(\hat J\otimes J)W(\hat J\otimes J)(F_1^*\otimes 1)\Lambda(\overline{A^{-\frac12}yA^{\frac12}})\otimes\Lambda(\overline{xA^{\frac12}})\\
&\quad \text{(because}\quad W^*=(\hat J\otimes J)W(\hat J\otimes J))\\
&= (\hat J\otimes J)\left(1\otimes\sigma_{-\frac{i}{2}}(F_2)\right)W\left(R(F_1)\otimes 1\right)(\hat J\otimes J)A^{-\frac12}\Lambda_\Omega(y)\otimes\Lambda_\Omega(x)\\
&\quad \text{(because}\quad R(x)=\hat J x^*\hat J,\quad\text{and}\quad \Lambda(\overline{A^{-\frac12}yA^{\frac12}})=A^{-\frac12}\Lambda_\Omega(y)\ \text{by Proposition 18)}\\
&= (\hat J\otimes J)W\Delta\left(\sigma_{-\frac{i}{2}}(F_2)\right)\left(R(F_1)\otimes 1\right)(\hat J\otimes J)A^{-\frac12}\Lambda_\Omega(y)\otimes\Lambda_\Omega(x)\\
&\quad \text{(because}\quad \Delta(x)=W^*(1\otimes x)W).
\end{aligned}
$$

So we just have to compute:

$$\Delta\left(\sigma_{-\frac{i}{2}}(F_2)\right)\left(R(F_1)\otimes 1\right)(g,h) = F_1(g^{-1})\sigma_{-\frac{i}{2}}(F_2)(gh)$$

$$= (\iota\otimes\sigma_{-\frac{i}{2}})(F)(g^{-1},gh) = (\widetilde{\iota\otimes\sigma_{-\frac{i}{2}}})(F)(g,h).$$

■ The next lemma is necessary to finish the proof of the theorem.

Lemma 6

(1) We have $\hat J A^{-\frac12} = A^{\frac12}\hat J$.

(2) The operator $(\iota\otimes\sigma_{-\frac{i}{2}})(\tilde\Omega)$ *is normal, affiliated with* $M\otimes M$*, and its polar decomposition is*

$$(\iota\otimes\sigma_{-\frac{i}{2}})(\tilde\Omega) = \tilde\Omega(A^{-\frac12}\otimes 1).$$

Proof. Let $\alpha\in\mathbb{R}$ and $\mathcal{D}_\alpha$ the horizontal strip bounded by $\mathbb{R}$ and $\mathbb{R}-i\alpha$.

(1) Let $\xi\in\mathcal{D}(A^{-\frac12})$. There exists a continuous bounded extension F of $t\mapsto A^{it}\xi$ on $\mathcal{D}_{-\frac12}$ which is analytic on $\mathcal{D}^0_{-\frac12}$. The function $G(z)=\hat J F(\overline{z})$ is continuous bounded on $\mathcal{D}_{\frac12}$ and analytic on $\mathcal{D}^0_{\frac12}$. Moreover :

$$R(A^{-it})(g) = \Psi(g^{-1},\gamma_t) = \Psi(g,\gamma_t^{-1}) = A^{it}(g),\quad \text{for all } g\in G, t\in\mathbb{R}.$$

Thus, $\hat J A^{it}\hat J = R(A^{-it}) = A^{it}$. We deduce $G(t)=\hat J A^{it}\xi = A^{it}\hat J\xi$. This means that $\hat J\xi\in\mathcal{D}(A^{\frac12})$ and $A^{\frac12}\hat J\xi = G(-\frac{i}{2}) = \hat J F(\frac{i}{2}) = \hat J A^{-\frac12}\xi$, so $\hat J A^{-\frac12}\subset A^{\frac12}\hat J$. The other inclusion can be proved in the same way.

(2) Note that

$$(\iota\otimes\sigma_t)(\tilde\Omega)(g,h) = \Psi(g^{-1},gh\gamma_t^{-1}) = \Psi(g^{-1},gh)\Psi(g,\gamma_t) = \tilde\Omega(A^{-it}\otimes 1)(g,h).$$

We conclude the proof applying Proposition 1. ■

We can now prove the theorem. Let $x \in N$ and $y \in L$. Put $\xi = \hat{J}\Lambda_\Omega(y) \in \mathcal{D}(A^{\frac{1}{2}})$ and $\eta = J\Lambda_\Omega(x)$. By Lemma 6 (2), $A^{\frac{1}{2}}\xi \otimes \eta \in \mathcal{D}\left((\iota \otimes \sigma_{-\frac{i}{2}})(\tilde{\Omega})\right)$. Thus, using Lemma 1 (1), there exists $\tilde{\Omega}_n \in L^\infty(G \times G) \cap \mathcal{D}(\iota \otimes \sigma_{-\frac{i}{2}})$ such that

$$\tilde{\Omega}_n \to \tilde{\Omega} \quad \sigma\text{-strongly}* \quad \text{and} \quad (\iota \otimes \sigma_{-\frac{i}{2}})(\tilde{\Omega}_n)(A^{\frac{1}{2}}\xi \otimes \eta) \to (\iota \otimes \sigma_{-\frac{i}{2}})(\tilde{\Omega})(A^{\frac{1}{2}}\xi \otimes \eta).$$

Because $\tilde{\tilde{F}} = F$, we also have $\Omega_n \to \Omega$ σ-strongly*, so

$$\Delta(x)\Omega_n^*(y \otimes 1) \to \Delta(x)\Omega^* y \otimes 1 \quad \sigma\text{-strongly}^*.$$

By Lemma 5, $\Delta(x)\Omega_n^*(y \otimes 1) \in \mathcal{N}_{\varphi_\Omega \otimes \varphi_\Omega}$ and

$$\begin{aligned}
(\Lambda_\Omega \otimes \Lambda_\Omega)\left(\Delta(x)\Omega_n^*(y \otimes 1)\right) &= (\hat{J} \otimes J)W(\iota \otimes \sigma_{-\frac{i}{2}})(\tilde{\Omega}_n)(\hat{J} \otimes J)\left(A^{-\frac{1}{2}}\Lambda_\Omega(y) \otimes \Lambda_\Omega(x)\right) \\
&= (\hat{J} \otimes J)W(\iota \otimes \sigma_{-\frac{i}{2}})(\tilde{\Omega}_n)(A^{\frac{1}{2}}\xi \otimes \eta) \quad \text{(by Lemma 6(1))} \\
&\to (\hat{J} \otimes J)W(\iota \otimes \sigma_{-\frac{i}{2}})(\tilde{\Omega})(A^{\frac{1}{2}}\xi \otimes \eta) \\
&= (\hat{J} \otimes J)W\tilde{\Omega}(\xi \otimes \eta) \quad \text{(by Lemma 6(2))} \\
&= (\hat{J} \otimes J)W\tilde{\Omega}(\hat{J} \otimes J)(\Lambda_\Omega(y) \otimes \Lambda_\Omega(x)).
\end{aligned}$$

Because $\Lambda_\Omega \otimes \Lambda_\Omega$ is σ-strong* - norm closed, we have $\Delta(x)\Omega^*(y \otimes 1) \in \mathcal{N}_{\varphi_\Omega \otimes \varphi_\Omega}$, so $\Delta_\Omega(x)(y \otimes 1) \in \mathcal{N}_{\varphi_\Omega \otimes \varphi_\Omega}$ and

$$\begin{aligned}
(\Lambda_\Omega \otimes \Lambda_\Omega)\left(\Delta_\Omega(x)(y \otimes 1)\right) &= \Omega\left(\Lambda_\Omega \otimes \Lambda_\Omega\right)\left(\Delta(x)\Omega^*(y \otimes 1)\right) \\
&= \Omega(\hat{J} \otimes J)W\tilde{\Omega}(\hat{J} \otimes J)(\Lambda_\Omega(y) \otimes \Lambda_\Omega(x)) \\
&= W_\Omega^*(\Lambda_\Omega(y) \otimes \Lambda_\Omega(x)).
\end{aligned}$$

■

Let $R_\Omega = uR(x)u^*$ be the *-anti-automorphism of M, where $u = \alpha(\Psi(\cdot^{-1}, \cdot))$.

Proof of Theorem1. Let $x, y \in \mathcal{N}_{\varphi_\Omega}$. By Theorem5, we have

$$\begin{aligned}
&||\left(\Lambda_\Omega \otimes \Lambda_\Omega\right)\left(\Delta_\Omega(x)y \otimes 1\right)||^2 = ||\Lambda_\Omega(y) \otimes \Lambda_\Omega(x)||^2 \\
&\Leftrightarrow \left(\omega_{\Lambda_\Omega(y)} \otimes \varphi_\Omega\right)\left(\Delta_\Omega(x^*x)\right) = \omega_{\Lambda_\Omega(y)}(1)\varphi_\Omega(x^*x). \qquad (6)
\end{aligned}$$

Let $\omega \in M_*$, $\omega \geq 0$. The inclusion $M \subset \mathcal{B}(H)$ is standard, so there is $\xi \in H$ such that $\omega = \omega_\xi$. Let $a_i \in M$ such that $\Lambda_\Omega(a_i) \to \xi$. Then

$$\omega_{\Lambda_\Omega(a_i)}(x) \to \omega(x), \quad \text{for all} \quad x \in M. \qquad (7)$$

To show that φ_Ω is left invariant, it suffices to show that $\Delta_\Omega(x^*x) \in \mathcal{N}_{\iota \otimes \varphi_\Omega}$ when $x \in \mathcal{N}_{\varphi_\Omega}$. Indeed, in this case we have, using (7),

$$\begin{aligned}
\omega_{\Lambda_\Omega(a_i)}(1)\varphi_\Omega(x^*x) \to \omega(1)\varphi_\Omega(x^*x) \quad \text{and,} \\
\left(\omega_{\Lambda_\Omega(a_i)} \otimes \varphi_\Omega\right)\left(\Delta_\Omega(x^*x)\right) \to \left(\omega \otimes \varphi_\Omega\right)\left(\Delta_\Omega(x^*x)\right).
\end{aligned}$$

This implies, using (6), that for all $\omega \in M_*^+$ and $x \in \mathcal{N}_{\varphi_\Omega}$,

$$(\omega \otimes \varphi_\Omega)(\Delta_\Omega(x^*x)) = \omega(1)\varphi_\Omega(x^*x),$$

i.e., φ_Ω is left invariant. Let us show that $\Delta_\Omega(x^*x) \in \mathcal{N}_{\iota\otimes\varphi_\Omega}$. We put

$$m = (\iota \otimes \varphi_\Omega)(\Delta_\Omega(x^*x)) \in M_+^{\text{ext}}.$$

The spectral decomposition of m is $m = \int_0^\infty \lambda de_\lambda + \infty\,.\,p$. From (6) we see that, for all $y \in \mathcal{N}_{\varphi_\Omega}$, $m(\omega_{\Lambda_\Omega(y)}) < \infty$. Thus, the set $\{\omega \in M_*^+ \mid m(\omega) < \infty\}$ is dense in M_*^+. This implies $p = 0$ and $m = m_T$, where T is the positive operator affiliated with M defined by

$$T = \int_0^\infty \lambda de_\lambda.$$

So, we only have to show that T is a bounded operator. Using again (6) and the definition of m_T, we see that, for all $y \in \mathcal{N}_{\varphi_\Omega}$, $\Lambda_\Omega(y) \in \mathcal{D}(A^{\frac{1}{2}})$ and

$$||T^{\frac{1}{2}}\Lambda_\Omega(y)||^2 = \varphi_\Omega(x^*x)||\Lambda_\Omega(y)||^2.$$

Thus, T is a bounded operator.

It is easy to check (see [16]) that $\Delta_\Omega \circ R_\Omega = \sigma(R_\Omega \otimes R_\Omega)\Delta_\Omega$, so the right invariance of $\varphi_\Omega \circ R_\Omega$ follows. Thus, (M, Δ_Ω) is a l.c. quantum group and it follows immediately from Theorem 5 that W_Ω is its multiplicative unitary. Our next aim is to show that $\psi_\Omega = \varphi_\Omega \circ R_\Omega$. We compute:

$$\begin{aligned} R\left(u\sigma^\Omega_{-t}(u^*)\right)(g) &= u(g^{-1})u^*(g^{-1}\gamma_t) = u(g)\Psi(g^{-1}\gamma_t, g^{-1}\gamma_t) \\ &= u(g)u^*(g)\Psi(g^{-1},\gamma_t)\Psi(\gamma_t, g^{-1})\Psi(\gamma_t,\gamma_t) \\ &= \lambda^{it^2}(A^{it}B^{it})(g). \end{aligned}$$

This implies

$$\begin{aligned} [D\varphi_\Omega \circ R_\Omega : D\psi]_t &= [D(\varphi_\Omega)_u \circ R : D\varphi \circ R]_t = R\left([D(\varphi_\Omega)_u : D\varphi]^*_{-t}\right) \\ &= R\left([D(\varphi_\Omega)_u : D\varphi_\Omega]^*_{-t}\right) R\left([D\varphi_\Omega : D\varphi]^*_{-t}\right) \\ &= R\left(u\sigma^\Omega_{-t}(u^*)\right) R\left(\lambda^{-i\frac{t^2}{2}}A^{it}\right) \\ &= \lambda^{it^2}A^{it}B^{it}(\lambda^{-i\frac{t^2}{2}}A^{-it}) \\ &= \lambda^{i\frac{t^2}{2}}B^{it}. \end{aligned}$$

Thus, $\psi_\Omega = \varphi_\Omega \circ R_\Omega$. In order to finish the proof, we have to compute the scaling group and the scaling constant. Recall that if (M, Δ) is a l.c. quantum group, then the scaling group is the unique one-parameter group τ_t on M such that $\Delta \circ \sigma_t = (\tau_t \otimes \sigma_t) \circ \Delta$. Since $(\iota \otimes \sigma_t)(\Omega) = \Omega(A^{it} \otimes 1)$, using $\tau_t \circ \alpha = \alpha$, we have $(\tau_t \otimes \sigma_t)(\Omega) = \Omega(A^{it} \otimes 1)$, which gives:

$$\begin{aligned}
(\tau_t \otimes \sigma_t^{\Omega})(\Delta_\Omega(x)) &= (1 \otimes A^{it})(\tau_t \otimes \sigma_t)(\Omega)(\tau_t \otimes \sigma_t)(\Delta(x))(\tau_t \otimes \sigma_t)(\Omega^*)(1 \otimes A^{-it}) \\
&= \Omega(A^{it} \otimes A^{it})(\tau_t \otimes \sigma_t)(\Delta(x))(A^{-it} \otimes A^{-it})\Omega^* \\
&= \Omega\Delta(A^{it})\Delta(\sigma_t(x))\Delta(A^{-it})\Omega^* \\
&= \Delta_\Omega(\sigma_t^{\Omega}(x)).
\end{aligned}$$

This relation characterizes the scaling group of (M, Δ_Ω). Recall that the scaling constant of (M, Δ) verifies $\varphi \circ \tau_t = \nu^{-t}\varphi$. Because $\tau_t(A^{is}) = A^{is}$, for all $t, s \in \mathbb{R}$, we deduce that $\varphi_\Omega \circ \tau_t^{\Omega} = \varphi_\Omega \circ \tau_t = \nu^{-t}\varphi_\Omega$. Thus, $\nu^{\Omega} = \nu$.

Let us denote by X and Y the operators

$$X = \Omega^* \quad \text{and} \quad Y = (\hat{J} \otimes J)(u^* \otimes 1)\Omega(\hat{J} \otimes J).$$

Note that $\tilde{\Psi}^*(g,h) = \Psi^*(g^{-1}, g)\Psi(g,h)$, so $\tilde{\Omega}^* = (u^* \otimes 1)\Omega$ and

$$W_\Omega = (\hat{J} \otimes J)\tilde{\Omega}^* W^* (\hat{J} \otimes J)\Omega^* = (\hat{J} \otimes J)\tilde{\Omega}^*(\hat{J} \otimes J)W\Omega^* = YWX.$$

From now on we suppose that G is abelian, we switch to the additive notations for its operations and denote by $\hat{G}$ its dual. Recall that the notations u_γ, L_γ and R_γ where introduced in Section 2.7. Note that $R(L_\gamma) = L_\gamma^* = L_{-\gamma}$.

Proposition 4 *R_Ω is the unitary antipode of (M, Δ_Ω). Moreover,*

- $\delta_\Omega = \delta A^{-1} B$,
- $\mathcal{D}(S_\Omega) = \mathcal{D}(S)$ *and, for all* $x \in \mathcal{D}(S)$, $S_\Omega(x) = uS(x)u^*$.

Proof. If (M, Δ) is a l.c. quantum group, then the unitary antipode is the unique *-anti-automorphism R of M such that $R((\iota \otimes \omega_{\xi,\eta})(W)) = (\iota \otimes \omega_{J\eta,J\xi})(W)$. Let us define two *-homomorphisms by

$$\begin{aligned}
\pi' &: L^\infty(G \times G) \to M \otimes M' : \pi'(F) = (\hat{J} \otimes J)(\alpha \otimes \alpha)(F)^*(\hat{J} \otimes J), \\
\pi &: L^\infty(G \times G) \to M \otimes M : \pi(F) = (\alpha \otimes \alpha)(F).
\end{aligned}$$

We want to prove that, for all $F, G \in L^\infty(G \times G)$ and $\xi, \eta \in H$,

$$R\left((\iota \otimes \omega_{\xi,\eta})\left(\pi'(F)W\pi(G)\right)\right) = (\iota \otimes \omega_{J\eta,J\xi})\left(\pi'(G)W\pi(F)\right). \tag{8}$$

By linearity and continuity, it suffices to prove (8) for $F = u_{\gamma_1} \otimes u_{\gamma_2}$ and $G = u_{\gamma_2} \otimes u_{\gamma_4}$ with $\gamma_i \in \hat{G}$. We have

$$\pi'(u_{\gamma_1} \otimes u_{\gamma_2}) = L_{-\gamma_1} \otimes R_{-\gamma_2} \quad \text{and} \quad \pi(u_{\gamma_3} \otimes u_{\gamma_4}) = L_{\gamma_3} \otimes L_{\gamma_4}, \quad \text{so}$$

$$
\begin{aligned}
& R\left((\iota\otimes\omega_{\xi,\eta})\left(\pi^{'}(u_{\gamma_1}\otimes u_{\gamma_2})W\pi(u_{\gamma_3}\otimes u_{\gamma_4})\right)\right) \\
= \quad & R\left((\iota\otimes\omega_{\xi,\eta})\left(L_{-\gamma_1}\otimes R_{-\gamma_2}WL_{\gamma_3}\otimes L_{\gamma_4}\right)\right) \\
= \quad & R\left(L_{-\gamma_1}(\iota\otimes L_{\gamma_4}.\omega_{\xi,\eta}.R_{-\gamma_2})\left(W\right)L_{\gamma_3}\right) \\
= \quad & L_{-\gamma_3}R\left((\iota\otimes\omega_{L_{\gamma_4}\xi,R_{\gamma_2}\eta})\left(W\right)\right)L_{\gamma_1} \\
= \quad & L_{-\gamma_3}(\iota\otimes\omega_{JR_{\gamma_2}\eta,JL_{\gamma_4}\xi})\left(W\right)L_{\gamma_1} \\
= \quad & L_{-\gamma_3}(\iota\otimes\omega_{L_{\gamma_2}J\eta,R_{\gamma_4}J\xi})\left(W\right)L_{\gamma_1} \\
= \quad & (\iota\otimes L_{\gamma_2}.\omega_{J\eta,J\xi}.R_{-\gamma_4})\left(L_{-\gamma_3}\otimes 1WL_{\gamma_1}\otimes 1\right) \\
= \quad & (\iota\otimes\omega_{J\eta,J\xi})\left(L_{-\gamma_3}\otimes R_{-\gamma_4}WL_{\gamma_1}\otimes L_{\gamma_2}\right) \\
= \quad & (\iota\otimes\omega_{J\eta,J\xi})\left(\pi^{'}(u_{\gamma_3}\otimes u_{\gamma_4})W\pi(u_{\gamma_1}\otimes u_{\gamma_2})\right).
\end{aligned}
$$

Note that $Y=\pi^{'}(\tilde{\Psi})$, $X=\pi(\Psi^*)$ and $\pi(\tilde{\Psi})(u^*\otimes 1)=\tilde{\Omega}(u^*\otimes 1)=\Omega^*=X$. Also, using $R(u^*)=u^*$, we have

$$
\begin{aligned}
(u\otimes 1)\pi^{'}(\Psi^*) &= (u\otimes 1)(\hat{J}\otimes J)\Omega(\hat{J}\otimes J) \\
&= (\hat{J}\otimes J)(R(u^*)\otimes 1)\Omega(\hat{J}\otimes J) \\
&= (\hat{J}\otimes J)(u^*\otimes 1)\Omega(\hat{J}\otimes J) \\
&= (\hat{J}\otimes J)\tilde{\Omega}^*(\hat{J}\otimes J)=Y.
\end{aligned}
$$

Using these remarks and relation (8), one has

$$
\begin{aligned}
R_\Omega\left((\iota\otimes\omega_{\xi,\eta})\left(W_\Omega\right)\right) &= uR\left((\iota\otimes\omega_{\xi,\eta})\left(\pi^{'}(\tilde{\Psi})W\pi(\Psi^*)\right)\right)u^* \\
&= (\iota\otimes\omega_{J\eta,J\xi})\left((u\otimes 1)\pi^{'}(\Psi^*)W\pi(\tilde{\Psi})(u^*\otimes 1)\right) \\
&= (\iota\otimes\omega_{J\eta,J\xi})\left(YWX\right) \\
&= (\iota\otimes\omega_{J_\Omega\eta,J_\Omega\xi})\left(W_\Omega\right).
\end{aligned}
$$

Where we use, in the last equality, the fact that $J_\Omega=\lambda^{\frac{i}{8}}J$ so $\omega_{J_\Omega\eta,J_\Omega\xi}=\omega_{J\eta,J\xi}$. This relation characterizes the unitary antipode of (M,Δ_Ω). We have

$$
\begin{aligned}
[D\psi_\Omega : D\varphi_\Omega]_t &= [D\psi_\Omega : D\psi]_t[D\psi : D\varphi]_t[D\varphi : D\varphi_\Omega]_t \\
&= (\lambda^{i\frac{t^2}{2}}B^{it})(\nu^{i\frac{t^2}{2}}\delta^{it})(\lambda^{-i\frac{t^2}{2}}A^{-it}) \\
&= \nu^{i\frac{t^2}{2}}(\delta A^{-1}B)^{it}.
\end{aligned}
$$

Thus, $\delta_\Omega=\delta A^{-1}B$ (because we have seen in the proof of Theorem 1 that $\psi_\Omega=\varphi_\Omega\circ R_\Omega$). The last statement is clear. ■

Remark. If (M,Δ) is a Kac algebra, (M,Δ_Ω) is not in general a Kac algebra (see Section 5.1). However, in the case considered in [3], [16], and [6], when

$\alpha(L^\infty(G))$ belongs to the fixed point subalgebra of M with respect to σ_t, then γ_t is trivial and we have $A^{-1}B = 1$, so (M, Δ_Ω) is a Kac algebra .

Remark. The map $L^\infty(G \times G) \to \mathcal{B}(H \otimes H)\ :\ F \mapsto \pi^{'}(\tilde{F})W\pi(F^*)$ is σ-strong*-σ-weak continuous. So, if $(x \mapsto \Psi_x)$ is σ-strongly* continuous map from $\mathbb{R}$ to $L^\infty(G \times G)$ such that Ψ_x is a continuous bicharacter, for all $x \in \mathbb{R}$, then, denoting by W_x the multiplicative unitary of the twisted l.c. quantum group associated with Ψ_x, the map $x \mapsto W_x$ from $\mathbb{R}$ to the unitaries of $\mathcal{B}(H \otimes H)$ is σ-weakly continuous. This is the case for the example of section 5.1 and for the examples constructed in [5] and [6].

4 Rieffel's deformations of locally compact quantum group

This section is devoted to the proof of Theorem 2. We use the hypotheses and notations from the previous section and from Section 2.7. So let $\widehat{G} < (M, \Delta)$ be a stable co-subgroup with G abelian. Recall that we have (see Section 2.8) two unitary representations of $\hat{G}$: $\gamma \mapsto L_\gamma$ and $\gamma \mapsto R_\gamma$ of $\widehat{G}$. This gives two *-homomorphisms from $L^\infty(G)$ to $\mathcal{B}(H)$, π_L and π_R, respectively. We have

$$\pi_L = \alpha \quad \text{and} \quad \pi_R(F) = J\alpha(\overline{F(-\cdot)})J = J\widehat{J}\alpha(F)\widehat{J}J.$$

Recall that $W_\Omega = YWX$, where $X = (\alpha \otimes \alpha)(\Psi^*) = (\pi_L \otimes \pi_L)(\Psi^*)$ and $Y = (\widehat{J} \otimes J)\tilde{\Omega}^*(\widehat{J} \otimes J) = (\pi_L \otimes \pi_R)(\tilde{\Psi}) = (\alpha(u) \otimes 1)(\pi_L \otimes \pi_R)(\Psi^*)$. Note that $\widehat{G} < (M_\Omega, \Delta_\Omega)$ is also stable (by the preceding section), so the results of section 2.8 can be applied also to $\widehat{G} < (M_\Omega, \Delta_\Omega)$. Thus, we have a left-right action α of $\widehat{G}^2$ on $\widehat{M}$ and also a left-right action β of $\widehat{G}^2$ on $\widehat{M_\Omega}$. We denote by the same π the canonical morphism from $\widehat{M}$ in the crossed product $N = \widehat{G}^2 \ltimes \widehat{M}$ and from $\widehat{M_\Omega}$ in $\widehat{G}^2 \ltimes \widehat{M_\Omega}$. Also we denote by $\lambda_{\gamma_1,\gamma_2}$ the canonical unitaries in the two crossed products and by the same θ the dual action on $\widehat{G}^2 \ltimes \widehat{M}$ and $\widehat{G}^2 \ltimes \widehat{M_\Omega}$. Recall that θ and λ verify

$$\theta_{g_1,g_2}(\lambda_{\gamma_1,\gamma_2}) = \overline{<\gamma_1, g_1><\gamma_2, g_2>}\lambda_{\gamma_1,\gamma_2}.$$

The unitary representations $\gamma \mapsto \lambda_{(\gamma,0)}$, $\gamma \mapsto \lambda_{(0,\gamma)}$ and λ give unital normal *-homomorphisms $\lambda_L, \lambda_R : L^\infty(G) \to \widehat{G}^2 \ltimes \widehat{M}$ and $\lambda : L^\infty(G^2) \to \widehat{G}^2 \ltimes \widehat{M}$ verifying

$$\lambda_L(u_\gamma) = \lambda_{(\gamma,0)}, \quad \lambda_R(u_\gamma) = \lambda_{(0,\gamma)}, \quad \lambda(u_{(\gamma_1,\gamma_2)}) = \lambda_{\gamma_1,\gamma_2}.$$

Since $\lambda(f_1 \otimes f_2) = \lambda_L(f_1)\lambda_R(f_2)$, then

$$\begin{aligned} \theta_{(g_1,g_2)}(\lambda_L(F)) &= \lambda_L(F(\cdot - g_1)), \quad \text{for any } F \in L^\infty(G). \qquad (9)\\ \theta_{(g_1,g_2)}(\lambda_R(F)) &= \lambda_R(F(\cdot - g_2)), \quad \text{for any } F \in L^\infty(G). \end{aligned}$$

We have for the twisted dual action θ^Ψ:

$$\theta^\Psi_{(g_1,g_2)}(\pi(x)) = \pi(\alpha_{(\Psi_{-g_1},\Psi_{g_2})}(x)), \quad \text{for all } x \in \widehat{M}. \qquad (10)$$

Considering the following unitaries in $M \otimes N$:

$\tilde{X} = (\alpha \otimes \lambda_L)(\Psi^*)$, $\tilde{Y} = (\alpha \otimes \lambda_R)(\tilde{\Psi}) = (\alpha(u) \otimes 1)(\alpha \otimes \lambda_R)(\Psi^*)$, $\tilde{W} = (\iota \otimes \pi)(W)$,

we put $\tilde{W}_\Omega = \tilde{Y}\tilde{W}\tilde{X}$. Let N_Ω be the fixed point subalgebra of $\widehat{G}^2 \ltimes \widehat{M}$ under the twisted dual action. The step to prove Theorem 2 is to show that $\widehat{M_\Omega}$ is isomorphic to N_Ω, for this we need a preliminary lemma. Let $\mathcal{B}$ be the von Neumann algebra acting on H and generated by $\{(\omega \otimes \iota)(W\Omega^*) \,|\, \omega \in \mathcal{B}(H)_*\}$.

Lemma 7 *We have:*

- $\mathcal{B} \vee \{L_\gamma \,|\, \gamma \in \widehat{G}\}'' = \widehat{M} \vee \{L_\gamma \,|\, \gamma \in \widehat{G}\}''$,
- $\mathcal{B} \vee \{R_\gamma \,|\, \gamma \in \widehat{G}\}'' = \widehat{M_\Omega} \vee \{R_\gamma \,|\, \gamma \in \widehat{G}\}''$.

Proof. First, take a net in the vector space spanned by elements $u_{\gamma_1} \otimes u_{\gamma_2}$ such that $\sum c_{i,j} u_{\gamma_i,\gamma_j} \to \Psi$ strongly*. Then $(\omega \otimes \iota)(W\Omega^*)$ is the weak limit of $\sum \overline{c_{i,j}}(L_{-\gamma_i}.\omega \otimes \iota)(W)L_{-\gamma_j}$, so $\mathcal{B} \subset \widehat{M} \vee \{L_\gamma \,|\, \gamma \in \widehat{G}\}''$. For the converse inclusion note that $(\omega \otimes \iota)(W) = (\omega \otimes \iota)(W\Omega^*\Omega)$. Thus, $(\omega \otimes \iota)(W)$ is the weak limit of $\sum c_{i,j}(L_{\gamma_i}.\omega \otimes \iota)(W\Omega^*)L_{\gamma_j}$. The second assertion can be proved using the same technique. ■

Proposition 5 *There exists a *-isomorphism $\rho : \widehat{G}^2 \ltimes \widehat{M} \to \widehat{G}^2 \ltimes \widehat{M_\Omega}$ intertwining the actions θ^Ψ on $\widehat{G}^2 \ltimes \widehat{M}$ and θ on $\widehat{G}^2 \ltimes \widehat{M_\Omega}$. Moreover,*

$$\rho((\omega \otimes \iota)(\tilde{W}_\Omega)) = \pi((\omega \otimes \iota)(W_\Omega)).$$

Proof. Remark that if we put $V = (\mathcal{F} \otimes \mathcal{F})U$ where $\mathcal{F} :\to L^2(G)$ is the Fourier transform and $U : L^2(\widehat{G} \times \widehat{G}) \otimes H \to L^2(\widehat{G} \times \widehat{G}) \otimes H$ is the unitary defined by $(U\xi)(\gamma_1, \gamma_2) = L_{\gamma_1} R_{\gamma_2} \xi(\gamma_1, \gamma_2)$ then

$$\left\{ \begin{array}{lcl} V\pi(x)V^* & = & 1 \otimes 1 \otimes x, \\ V\lambda_{\gamma,0}V^* & = & u_\gamma \otimes 1 \otimes L_\gamma, \\ V\lambda_{0,\gamma}V^* & = & 1 \otimes u_\gamma \otimes R_\gamma. \end{array} \right.$$

Applying $\alpha \otimes \alpha \otimes \iota$, we conclude that the crossed products can be defined on $H \otimes H \otimes H$ by:

$$\begin{array}{rcl} \widehat{G}^2 \ltimes \widehat{M} & = & \{L_\gamma \otimes 1 \otimes L_\gamma \,|\, \gamma \in \widehat{G}\}'' \vee \{1 \otimes L_\gamma \otimes R_\gamma \,|\, \gamma \in \widehat{G}\}'' \vee 1 \otimes 1 \otimes \widehat{M}, \\ \widehat{G}^2 \ltimes \widehat{M_\Omega} & = & \{L_\gamma \otimes 1 \otimes L_\gamma \,|\, \gamma \in \widehat{G}\}'' \vee \{1 \otimes L_\gamma \otimes R_\gamma \,|\, \gamma \in \widehat{G}\}'' \vee 1 \otimes 1 \otimes \widehat{M_\Omega}. \end{array}$$

Put $\mathcal{W} = (\widehat{J} \otimes \widehat{J})W(\widehat{J} \otimes \widehat{J})$. Then $\mathcal{W}^*(1 \otimes x)\mathcal{W} = \Delta^{\mathrm{op}}(x)$, for all $x \in M$ and $[\mathcal{W}, 1 \otimes y] = 0$, for all $y \in \widehat{M}$. We have also $\mathcal{W}_\Omega = (\widehat{J_\Omega} \otimes \widehat{J_\Omega})W\Omega(\widehat{J_\Omega} \otimes \widehat{J_\Omega})$ with similar properties.

In the following computation we use the relations $\mathcal{W}^*(1 \otimes L_\gamma)\mathcal{W} = L_\gamma \otimes L_\gamma$, $\mathcal{W}(1 \otimes R_\gamma)\mathcal{W}^* = L_\gamma \otimes R_\gamma$ and similar relations with $\mathcal{W}_\Omega$. We use also the equality

$[\mathcal{W}_{13}\Omega_{31}^*, W_{23}\Omega_{23}^*] = 0$ implying $[\mathcal{W}\Omega_{21}^*, 1 \otimes y] = 0$, for all $y \in \mathcal{B}$. Finally, using Lemma 7, we have:

$$
\begin{array}{c}
\widehat{G}^2 \ltimes \widehat{M} = \{L_\gamma \otimes 1 \otimes L_\gamma \,|\, \gamma \in \widehat{G}\}'' \vee \{1 \otimes L_\gamma \otimes R_\gamma \,|\, \gamma \in \widehat{G}\}'' \vee 1 \otimes 1 \otimes \widehat{M} \\
\downarrow \ \mathrm{Ad}(\mathcal{W}_{13}) \\
\begin{array}{rll}
 & \{1 \otimes 1 \otimes L_\gamma\}'' \vee \{L_\gamma \otimes L_\gamma \otimes R_\gamma\}'' \vee 1 \otimes 1 \otimes \widehat{M} & \\
= & \{1 \otimes 1 \otimes L_\gamma\}'' \vee \{L_\gamma \otimes L_\gamma \otimes R_\gamma\}'' \vee 1 \otimes 1 \otimes \widehat{B} & := L_1
\end{array} \\
\downarrow \ \mathrm{Ad}(\Omega_{32}\mathcal{W}_{23}^*\Omega_{31}\mathcal{W}_{13}^*) \\
\begin{array}{rll}
 & \{L_\gamma \otimes L_\gamma \otimes L_\gamma\}'' \vee \{1 \otimes 1 \otimes R_\gamma\}'' \vee 1 \otimes 1 \otimes \widehat{B} & \\
= & \{L_\gamma \otimes L_\gamma \otimes L_\gamma\}'' \vee \{1 \otimes 1 \otimes R_\gamma\}'' \vee 1 \otimes 1 \otimes \widehat{M_\Omega} & := L_2
\end{array} \\
\downarrow \ \mathrm{Ad}((\mathcal{W}_\Omega)_{23}) \\
\{L_\gamma \otimes 1 \otimes L_\gamma \,|\, \gamma \in \widehat{G}\}'' \vee \{1 \otimes L_\gamma \otimes R_\gamma \,|\, \gamma \in \widehat{G}\}'' \vee 1 \otimes 1 \otimes \widehat{M_\Omega} \quad = \widehat{G}^2 \ltimes \widehat{M_\Omega}.
\end{array}
$$

Define $\rho := \rho_2 \circ \Phi \circ \rho_1$, where ρ_1, Φ and ρ_2 are the isomorphisms from $\widehat{G}^2 \ltimes \widehat{M}$ to L_1, from L_1 to L_2, and from L_2 to $\widehat{G}^2 \ltimes \widehat{M_\Omega}$, respectively. Then one can check that $\rho \circ \theta^\Psi_{g_1,g_2}(x) = \theta_{g_1,g_2} \circ \rho(x)$, for all $g_1, g_2 \in G$ and for all x of the form $\lambda_{\gamma_1,\gamma_2}$ (or $L_\gamma \otimes 1 \otimes L_\gamma$ and $1 \otimes L_\gamma \otimes R_\gamma$ in our description of the crossed products). Thus, to finish the proof we only have to show that $(\omega \otimes \iota)(\tilde{W}_\Omega) \in N_\Omega$ and $\rho((\omega \otimes \iota)(\tilde{W}_\Omega)) = \pi((\omega \otimes \iota)(W_\Omega))$. Using (9), one computes

$$
\begin{aligned}
(\iota \otimes \theta^\Psi_{(g_1,g_2)})(\tilde{X}) &= (\iota \otimes \theta_{(g_1,g_2)})(\tilde{X}) = (\alpha \otimes \lambda_L)(\Psi^*(.,. - g_1)) \\
&= (\alpha \otimes \lambda_L)(\Psi_{g_1} \otimes 1)(\alpha \otimes \lambda_L)(\Psi^*) \\
&= (\alpha(\Psi_{g_1}) \otimes 1)\tilde{X}. \qquad (11)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(\iota \otimes \theta^\Psi_{(g_1,g_2)})(\tilde{Y}) &= (\iota \otimes \theta_{(g_1,g_2)})(\tilde{Y}) = (\iota \otimes \theta_{(g_1,g_2)})((\alpha \otimes \lambda_R)(\tilde{\Psi})) \\
&= (\alpha \otimes \lambda_R)(\tilde{\Psi}(.,. - g_2)) = \tilde{Y}(\alpha(\Psi_{g_2}) \otimes 1). \qquad (12)
\end{aligned}
$$

And, using (10) and (4), one has

$$
\begin{aligned}
(\iota \otimes \theta^\Psi_{(g_1,g_2)})(\tilde{W}) &= (\iota \otimes \pi)\left((L_{\Psi_{-g_2}} \otimes 1)W(L_{\Psi_{-g_1}} \otimes 1)\right) \\
&= (\alpha(\Psi^*_{g_2}) \otimes 1)\tilde{W}(\alpha(\Psi^*_{g_1}) \otimes 1). \qquad (13)
\end{aligned}
$$

Now (11), (12), and (13) imply $(\iota \otimes \theta^\Psi_{(g_1,g_2)})(\tilde{W}_\Omega) = \tilde{W}_\Omega$, so $(\omega \otimes \iota)(\tilde{W}_\Omega) \in N_\Omega$. Now we want to show that $\rho((\omega \otimes \iota)(\tilde{W}_\Omega)) = \pi((\omega \otimes \iota)(W_\Omega))$. We take a net in the vector space spanned by elements $u_{\gamma_1} \otimes u_{\gamma_2}$ such that $\sum c_{i,j}(u_{\gamma_i} \otimes u_{\gamma_j}) \to \Psi$ strongly*, so $\sum \bar{c}_{i,j}(L_{-\gamma_i} \otimes \lambda_{-\gamma_j,0}) \to \tilde{X}$ and $\sum \bar{c}_{i,j}(\alpha(u) \otimes 1)(L_{-\gamma_i} \otimes \lambda_{0,-\gamma_j}) \to \tilde{Y}$ strongly*. This implies

$$
\sum \bar{c}_{i,j}\bar{c}_{k,l}\lambda_{0,-\gamma_j}\pi((L_{-\gamma_k}.\omega.L_{-\gamma_i}.\alpha(u) \otimes \iota)(W))\lambda_{-\gamma_l,0} \to (\omega \otimes \iota)(\tilde{W}_\Omega) \quad \text{weakly.}
$$

Thus $\rho_1((\omega\otimes\iota)(\tilde{W}_\Omega))$ is the weak limit of the net

$$\begin{aligned}&\sum \bar{c}_{i,j}\bar{c}_{k,l}(L_{-\gamma_j}\otimes L_{-\gamma_j}\otimes R_{-\gamma_j})(1\otimes 1\otimes (L_{-\gamma_k}.\omega.L_{-\gamma_i}.\alpha(u)\otimes\iota)(W))(1\otimes 1\otimes L_{-\gamma_l})\\ &=\sum_{i,j}\bar{c}_{i,j}(L_{-\gamma_j}\otimes L_{-\gamma_j}\otimes R_{-\gamma_j})(1\otimes 1\otimes(\omega.L_{-\gamma_i}.\alpha(u)\otimes\iota)(W\sum_{k,l}\bar{c}_{k,l}L_{-\gamma_k}\otimes L_{-\gamma_l}))\\ &\longrightarrow_{k,l}\sum_{i,j}\bar{c}_{i,j}(L_{-\gamma_j}\otimes L_{-\gamma_j}\otimes R_{-\gamma_j})(1\otimes 1\otimes(\omega.L_{-\gamma_i}.\alpha(u)\otimes\iota)(W\Omega^*)).\end{aligned}$$

and $\Phi\circ\rho_1((\omega\otimes\iota)(\tilde{W}_\Omega))$ is the weak limit of the net

$$\begin{aligned}&\sum\bar{c}_{i,j}(1\otimes 1\otimes R_{0,-\gamma_j})(1\otimes 1\otimes(\omega.L_{-\gamma_i}.\alpha(u)\otimes\iota)(W\Omega^*))\\ &=1\otimes 1\otimes(\omega\otimes\iota)(\sum\bar{c}_{i,j}(\alpha(u)\otimes 1)(L_{-\gamma_i}\otimes R_{-\gamma_j})W\Omega^*).\end{aligned}$$

Because $\sum\bar{c}_{i,j}(\alpha(u)\otimes 1)(L_{-\gamma_i}\otimes R_{-\gamma_j})\to Y$ weakly, we have

$$\Phi\circ\rho_1((\omega\otimes\iota)(\tilde{W}_\Omega))=1\otimes 1\otimes(\omega\otimes\iota)(W_\Omega).$$

This concludes the proof. ■

In particular, Proposition 5 implies that $N_\Omega=\{(\omega\otimes\iota)(\tilde{W}_\Omega)\,|\,\omega\in\mathcal{B}(H)_*\}''$ and that ρ is a *-isomorphism from N_Ω to $\widehat{M}_\Omega$ which sends $(\omega\otimes\iota)(\tilde{W}_\Omega)$ to $(\omega\otimes\iota)(W_\Omega)$. Thus, we can transport the l.c. quantum group structure from $\widehat{M}_\Omega$ to N_Ω. First, we show that the comultiplication introduced in Section 2.7 is the good one. For this we need

Proposition 6 *There exists a unique unital normal *-homomorphism* $\Gamma: N\to N\otimes N$ *such that*

$$\Gamma(\lambda_{\gamma_1,\gamma_2})=\lambda_{\gamma_1,0}\otimes\lambda_{0,\gamma_2}\quad\textit{and}\quad\Gamma(\pi(x))=(\pi\otimes\pi)\hat{\Delta}(x).$$

Proof. Like in the begining of the proof of Proposition 5 define the crossed product

$$\widehat{G}^2\ltimes\widehat{M}=\{L_\gamma\otimes 1\otimes L_\gamma\,|\,\gamma\in\widehat{G}\}''\vee\{1\otimes L_\gamma\otimes R_\gamma\,|\,\gamma\in\widehat{G}\}''\vee 1\otimes 1\otimes\widehat{M}.$$

Let $\mathcal{W}$ be the operator defined in the proof of Proposition 5 and Q be the unitary on $H\otimes H\otimes H\otimes H\otimes H\otimes H$ such that $Q^*=\Sigma_{45}\Sigma_{35}\mathcal{W}^*_{15}\hat{W}^*_{56}\Sigma_{45}$. We define $\Gamma(x)=Q^*(1\otimes x)Q$. Using that $\hat{W}^*(L_\gamma\otimes L_\gamma)\hat{W}=L_\gamma\otimes 1$, $\hat{\Delta}(x)=\hat{W}^*(1\otimes x)\hat{W}$, for all $x\in\hat{M}$, $[\hat{W},1\otimes y]$=0, for all $y\in M'$, $\mathcal{W}^*(1\otimes L_\gamma)\mathcal{W}=L_\gamma\otimes L_\gamma$ and $[\mathcal{W},1\otimes y]=0$, for all $y\in\hat{M}$, one can check that the needed properties of Γ. ■

The unitary $\Upsilon=(\lambda_R\otimes\lambda_L)(\tilde{\Psi}^*)\in N\otimes N$ allows to define the unital normal *-homomorphism $\Gamma_\Omega(x)=\Upsilon\Gamma(x)\Upsilon^*: N\to N\otimes N$ which is a comultiplication on N_Ω:

Proposition 7 *For all* $x\in N_\Omega$*, we have* $\Gamma_\Omega(x)\in N_\Omega\otimes N_\Omega$ *and*

$$(\rho\otimes\rho)(\Gamma_\Omega(x))=\hat{\Delta}_\Omega(\rho(x)).$$

Proof. It suffices to show that $(\iota\otimes\rho\otimes\rho)(\iota\otimes\Gamma_\Omega)(\tilde W_\Omega) = (W_\Omega)_{13}(W_\Omega)_{12}$. By the definition of Γ, one has, for any $F\in L^\infty(G)$,

$$\Gamma(\lambda_L(F)) = \lambda_L(F)\otimes 1 \quad\text{and}\quad \Gamma(\lambda_R(F)) = 1\otimes\lambda_R(F),$$

and since $1\otimes\Upsilon$ commutes with $\tilde X_{12}$ and with $\tilde Y_{13}$, one gets:

$$(\iota\otimes\Gamma_\Omega)(\tilde X) = \tilde X_{12} \quad\text{and}\quad (\iota\otimes\Gamma_\Omega)(\tilde Y) = \tilde Y_{13}.$$

Moreover,

$$\begin{aligned}
(\iota\otimes\Gamma_\Omega)(\tilde W) &= (1\otimes\Upsilon)(\iota\otimes\Gamma\circ\pi)(W)(1\otimes\Upsilon^*)\\
&= (1\otimes\Upsilon)(\iota\otimes\pi\otimes\pi)\Big((\iota\otimes\hat\Delta)(W)\Big)(1\otimes\Upsilon^*)\\
&= (1\otimes\Upsilon)(\iota\otimes\pi\otimes\pi)(W_{13}W_{12})(1\otimes\Upsilon^*)\\
&= \Upsilon_{23}\tilde W_{13}\tilde W_{12}\Upsilon^*_{23}.
\end{aligned}$$

Using (3), we can check the following relations on the generators u_γ of $L^\infty(G)$:

$$\begin{aligned}
&W(1\otimes\pi_R(F))W^* = (\pi_L\otimes\pi_R)(\Delta_G(F))\,,\\
&W^*(1\otimes\pi_L(F))W = (\pi_L\otimes\pi_L)(\Delta_G(F))\,, \text{for any } F\in L^\infty(G).
\end{aligned}$$

Then

$$\begin{aligned}
&W_{12}(\pi_R\otimes\pi_L)(\tilde\Psi^*)^*_{23}W^*_{12} = (\pi_L\otimes\pi_R\otimes\pi_L)\Big((\Delta_G\otimes\iota)(\tilde\Psi)\Big)\ ,\\
&W^*_{13}(\pi_R\otimes\pi_L)(\tilde\Psi^*)_{23}W_{13} = (\pi_L\otimes\pi_R\otimes\pi_L)\Big((\sigma\otimes\iota)\Big((\iota\otimes\Delta_G)(\tilde\Psi^*)\Big)\Big)\,.
\end{aligned}$$

Let us define

$$V = (\pi_L\otimes\pi_R\otimes\pi_L)\Big((\sigma\otimes\iota)\Big((\iota\otimes\Delta_G)(\tilde\Psi^*)\Big)\Big)(\pi_L\otimes\pi_R\otimes\pi_L)\Big((\Delta_G\otimes\iota)(\tilde\Psi)\Big)\,,$$

then we have

$$\begin{aligned}
(\iota\otimes\rho\otimes\rho)(\iota\otimes\Gamma_\Omega)(\tilde W_\Omega) &= (\iota\otimes\rho\otimes\rho)(\tilde Y_{13}\Upsilon_{23}\tilde W_{13}\tilde W_{12}\Upsilon^*_{23}\tilde X_{12})\\
&= Y_{13}W_{13}VW_{12}X_{12},
\end{aligned}$$

so it remains to calculate:

$$\begin{aligned}
&(\sigma\otimes\iota)\Big((\iota\otimes\Delta_G)(\tilde\Psi^*)\Big)(g,h,t)(\Delta_G\otimes\iota)(\tilde\Psi)(g,h,t)\\
&= (\iota\otimes\Delta_G)(\tilde\Psi^*)(h,g,t)(\Delta_G\otimes\iota)(\tilde\Psi)(g,h,t)\\
&= \Psi^*(-h,h+g+t)\Psi(-g-h,g+h+t) = \Psi(-g,g+h+t)\\
&= \Psi^*(g,t)\tilde\Psi(g,h).
\end{aligned}$$

Thus, $V = X_{13}Y_{12}$, and this concludes the proof. ■

Remark. One can show that α and β are actions of $\widehat{G}^2$ on the reduced dual C^*-algebras $\widehat{A}$ and $\widehat{A}_\Omega$. Moreover, the *-isomorphism ρ gives a *-isomorphism between the reduced crossed products $\widehat{G}^2 \ltimes \widehat{A}$ and $\widehat{G}^2 \ltimes \widehat{A}_\Omega$. So $\hat{A}$ is nuclear if and only if $\hat{A}_\Omega$ is nuclear. Moreover, the twisted dual action θ^Ψ gives a deformed $\widehat{G}^2$-product structure on $\widehat{G}^2 \ltimes \widehat{A}$ and the Landstad algebra for this $\widehat{G}^2$-product is $[(\omega \otimes \iota)(\tilde{W}_\Omega)]$, and it is isomorphic to $\widehat{A}_\Omega$. These results can be obtained directly from the universality property of crossed products (see [4]).

The rest of this section is devoted to the computation of the left invariant weight on $(N_\Omega, \Gamma_\Omega)$. Since $\rho : N = \hat{G}^2 \ltimes \hat{M} \to \hat{G}^2 \ltimes \hat{M}_\Omega$ is a $*$-isomorphism, one can consider two natural weights on N, $\varphi_1 = \tilde{\hat{\varphi}}$, the dual weight of $\hat{\varphi}$ on N, and $\varphi_2 = \tilde{\hat{\varphi}}_\Omega \circ \rho$, where $\tilde{\hat{\varphi}}_\Omega$ is the dual weight of $\hat{\varphi}_\Omega$ on $\hat{G}^2 \ltimes \hat{M}_\Omega$.

Lemma 8 *We have:*

1. $[D\hat{\varphi} \circ \alpha_{\gamma_1,\gamma_2} : D\hat{\varphi}]_t = \langle \gamma_2, \gamma_t \rangle = [D\hat{\varphi}_\Omega \circ \beta_{\gamma_1,\gamma_2} : D\hat{\varphi}_\Omega]_t$ $\forall t \in \mathbb{R}$, $\forall \gamma_1, \gamma_2 \in \widehat{G}$.

2. $[D\varphi_1 \circ \theta^\Psi_{g_1,g_2} : D\varphi_1]_t = \Psi(\gamma_t, g_2)$, *for all* $t \in \mathbb{R}$ *and all* $g_1, g_2 \in G$.

3. *For any n.s.f. weight* ν *on* N, ν *is invariant under the action* θ^Ψ *if and only if* $\theta^\Psi_{g_1,g_2}([D\nu : D\varphi_1]_t) = \Psi(\gamma_t, g_2)[D\nu : D\varphi_1]_t$.

Proof. Using Proposition 3(2), and because L_γ and R_γ are unitaries, we find $\hat{\varphi} \circ \alpha^L_\gamma = \hat{\varphi}, \quad \hat{\varphi} \circ \alpha^R_\gamma = \lambda(\gamma)\hat{\varphi}$, so

$$\begin{aligned} [D\hat{\varphi} \circ \alpha_{\gamma_1,\gamma_2} : D\hat{\varphi}]_t &= [D\hat{\varphi} \circ \alpha^L_{\gamma_1} \circ \alpha^R_{\gamma_2} : D\hat{\varphi} \circ \alpha^R_{\gamma_2}]_t [D\hat{\varphi} \circ \alpha^R_{\gamma_2} : D\hat{\varphi}]_t \\ &= \alpha^R_{-\gamma_2}\left([D\hat{\varphi} \circ \alpha^L_{\gamma_1} : D\hat{\varphi}]_t\right)[D\hat{\varphi} \circ \alpha^R_{\gamma_2} : D\hat{\varphi}]_t \\ &= \lambda(\gamma_2)^{it} = \langle \gamma_2, \gamma_t \rangle. \end{aligned}$$

The right-hand side of the first equality is obtained by considering the stable co-subgroup $\widehat{G} < (M, \Delta_\Omega)$. Let us prove the second assertion. Let $g_1, g_2 \in G$, define the unitary $v := \lambda_{(\Psi_{-g_1}, \Psi_{g_2})}$, and denote by $\varphi_1|_v$ the weight $\varphi_1|_v(x) = \varphi_1(vxv^*)$. Using the first assertion, we have

$$[D\varphi_1|_v : D\varphi_1]_t = v^* \sigma^1_t(v) = v^* \langle \Psi_{g_2}, \gamma_t \rangle v = \Psi(\gamma_t, g_2).$$

Note that $\varphi_1 \circ \theta^\Psi_{g_1,g_2} = \varphi_1|_v \circ \theta_{g_1,g_2}$, so

$$\begin{aligned} [D\varphi_1 \circ \theta^\Psi_{g_1,g_2} : D\varphi_1]_t &= [D\varphi_1|_v \circ \theta_{g_1,g_2} : D\varphi_1 \circ \theta_{g_1,g_2}]_t [D\varphi_1 \circ \theta_{g_1,g_2} : D\varphi_1]_t \\ &= \theta_{-g_1,-g_2}([D\varphi_1|_v : D\varphi_1]_t) = \Psi(\gamma_t, g_2). \end{aligned}$$

Putting $u_t = [D\nu : D\varphi_1]_t$ and using the second assertion, one has

$$[D\nu \circ \theta^\Psi_{g_1,g_2} : D\nu]_t = \theta^\Psi_{-g_1,-g_2}(u_t)[D\varphi_1 \circ \theta^\Psi_{g_1,g_2} : D\varphi_1]_t u^*_t = \theta^\Psi_{-g_1,-g_2}(u_t)\Psi(\gamma_t, g_2)u^*_t.$$

This concludes the proof. ■

Note that, using Lemma 8 (1), we have, for all $t \in \mathbb{R}, F \in L^\infty(G^2)$,

$$\sigma^1_t(\lambda(F)) = \lambda(F(\cdot, \cdot + \gamma_t)) = \sigma^2_t(\lambda(F)). \tag{14}$$

Let T be the strictly positive operator affiliated with N and such that $T^{it} = \lambda_R(\Psi(-\gamma_t, .))$. Using (14), we find $\sigma_t^1(T^{is}) = \lambda^{-its}T^{is}$, so one can consider the Vaes' weight $\tilde{\mu}_\Omega$ associated with T and λ^{-1}. This is the unique n.s.f. weight on N such that $[D\tilde{\mu}_\Omega : D\varphi_1]_t = \lambda^{\frac{-it^2}{2}}T^{it}$. From (9) we have $\theta^\Psi_{g_1,g_2}(T^{it}) = \lambda_R(\Psi(-\gamma_t, . - g_2) = \Psi(\gamma_t, g_2)T^{it}$. By Lemma 8(3), $\tilde{\mu}_\Omega$ is invariant under θ^Ψ,so the image $\tilde{\mu}_\Omega \circ \rho^{-1}$ of $\tilde{\mu}_\Omega$ in $\hat{G}^2 \ltimes \hat{M}_\Omega$ is invariant under the dual action. Thus, $\tilde{\mu}_\Omega \circ \rho^{-1}$ is the dual weight of some weight μ_Ω on $\hat{M}_\Omega$. To finish the proof of Theorem 2, we will show in Theorem 6 that $\mu_\Omega = \hat{\varphi}_\Omega$, for which we need

Proposition 8 *For all $t \in \mathbb{R}$ and all $x \in N$, we have*

$$\sigma_t^2(x) = T^{it}\sigma_t^1(x)T^{-it}.$$

Proof. By (14), it suffices to prove this equality for elements of the form $(\omega \otimes \iota)(\tilde{W}_\Omega)$. By definition of σ_t^2, we have

$$\sigma_t^2\left((\omega \otimes \iota)(\tilde{W}_\Omega)\right) = (\rho_t^\Omega(\omega) \otimes \iota)(\tilde{W}_\Omega),$$

where $\rho_t^\Omega(\omega)(x) = \omega(\delta_\Omega^{-it}\tau^\Omega_{-t}(x))$. Proposition 4 gives

$$\rho_t^\Omega(x) = \omega\left(\delta^{-it}A^{it}B^{-it}\tau_t(x)\right).$$

On the other hand, using (14), one has

$$(\iota \otimes \sigma_t^1)(\tilde{X}) = \tilde{X}, \quad (\iota \otimes \sigma_t^2)(\tilde{Y}) = (A^{it} \otimes 1)\tilde{Y},$$

which implies

$$\begin{aligned}
(\iota \otimes \sigma_t^1)(\tilde{W}_\Omega) &= (A^{it} \otimes 1)\tilde{Y}(\rho_t(\omega) \otimes \iota)(\tilde{W})\tilde{X} \\
&= (A^{it} \otimes 1)\tilde{Y}(\delta^{-it} \otimes 1)(\tau_{-t} \otimes \iota)(\tilde{W})\tilde{X} \\
&= (B^{it} \otimes 1)(\delta^{-it}A^{it}B^{-it} \otimes 1)(\tau_{-t} \otimes \iota)(\tilde{W}_\Omega) \\
&\quad \text{because } \delta^{it}\alpha(.)\delta^{-it} = \alpha(.) \text{ and } \tau_t \circ \alpha = \tau_t \\
&= (B^{it} \otimes 1)(\iota \otimes \sigma_t^2)(\tilde{W}_\Omega).
\end{aligned}$$

Next, using (3) with the character $\chi_t(g) = \Psi(\gamma_t, g)$, we find

$$\begin{aligned}
(B^{it} \otimes 1)\tilde{W}_\Omega &= \tilde{Y}(L_{-\chi_t} \otimes 1)\tilde{W}\tilde{X} = \tilde{Y}(\iota \otimes \pi)\left((L_{-\chi_t} \otimes 1)W\right)\tilde{X} \\
&= \tilde{Y}(\iota \otimes \pi)\left((1 \otimes R_{\chi_t})W(1 \otimes R^*_{\chi_t})\right)\tilde{X} = (1 \otimes \lambda_R(\chi_t))\tilde{W}_\Omega(1 \otimes \lambda_R(\chi_t)^*) \\
&= (1 \otimes T^{-it})\tilde{W}_\Omega(1 \otimes T^{it}). \qquad (15)
\end{aligned}$$

Thus, for all $t \in \mathbb{R}, \omega \in M_*$, one has

$$\begin{aligned}
\sigma_t^1((\omega \otimes \iota)(\tilde{W}_\Omega)) &= (\omega \otimes \iota)((B^{it} \otimes 1)(\iota \otimes \sigma_t^2)(\tilde{W}_\Omega)) \\
&= (\omega \otimes \iota)((\iota \otimes \sigma_t^2)((1 \otimes T^{-it})\tilde{W}_\Omega(1 \otimes T^{it}))) \\
&= T^{-it}\sigma_t^2((\omega \otimes \iota)(\tilde{W}_\Omega))T^{it},
\end{aligned}$$

where we used, in the last equation, $\sigma_t^2(T^{is}) = \lambda^{-ist}T^{is}$. ■

Theorem 6 *We have* $\mu_\Omega = \hat{\varphi}_\Omega$.

Let us denote by φ_P the Plancherel weight on $\mathcal{L}(\hat{G}^2)$, by Λ_P its canonical G.N.S. map, by $\lambda_L^{\hat{G}^2}$ and $\lambda_R^{\hat{G}^2}$ the $*$-homorphisms $L^\infty(G) \to \mathcal{L}(\hat{G}^2)$ coming from the representations $(\gamma \mapsto \lambda^{\hat{G}^2}_{(\gamma,0)})$ and $(\gamma \mapsto \lambda^{\hat{G}^2}_{(0,\gamma)})$, respectively, and by $T_1^{it} = \lambda_R^{\hat{G}^2}(\Psi(-\gamma_t, \cdot)$. Thus, $T = T_1 \otimes 1$. We also introduce the $*$-homomorphism $\alpha^{'}(F) = J\alpha(F)^*J$ and denote by $F \mapsto F^\circ$ the $*$-automorphism of $L^\infty(G \times G)$ defined by $F^\circ(g,h) = F(h, g+h)$.

The standard G.N.S. construction for φ_1 is $(L^2(\hat{G}^2, H), \iota, \Lambda_1)$, where a σ-strong-*-norm core for Λ_1 is given by

$$\mathcal{D}_1 = \left\{(x \otimes 1)(\omega \otimes \iota)(\tilde{W}) \,|\, x \in \mathcal{N}_{\varphi_P},\, \omega \in \mathcal{I}\right\},$$

and, if $x \in \mathcal{N}_{\varphi_P}$, $\omega \in \mathcal{I}$, we have

$$\Lambda_1\left((x \otimes 1)(\omega \otimes \iota)(\tilde{W})\right) = \Lambda_P(x) \otimes \xi(\omega).$$

Let $\lambda_\Omega(\omega)$, $\mathcal{I}_\Omega$, and ξ_Ω be the standard objects associated with (M, Δ_Ω). For φ_2, we take the G.N.S. construction $(L^2(\hat{G}^2, H), \tilde{\rho}, \Lambda_2)$, where a σ-strong-*-norm core for Λ_2 is

$$\mathcal{D}_2 = \left\{(x \otimes 1)(\omega \otimes \iota)(\tilde{W}_\Omega) \,|\, x \in \mathcal{N}_{\varphi_P},\, \omega \in \mathcal{I}_\Omega\right\},$$

and, if $x \in \mathcal{N}_{\varphi_P}$, $\omega \in \mathcal{I}_\Omega$, one has

$$\Lambda_2\left((x \otimes 1)(\omega \otimes \iota)(\tilde{W}_\Omega)\right) = \Lambda_P(x) \otimes \xi_\Omega(\omega).$$

Let us introduce the following sets:

$$\begin{aligned}
C_1 &= \left\{x \in \mathcal{N}_{\varphi_P} \,|\, T^{\frac{1}{2}}(x \otimes 1) \text{ is bounded}\right\},\\
C_1^0 &= \left\{x \in C_1 \,|\, \Lambda_P(x) \in \mathcal{D}(T_1^{-\frac{1}{2}})\right\},\\
C_2 &= \left\{\omega_{\xi,\eta} \in \mathcal{I}_\Omega \,|\, \eta \in \mathcal{D}(A^{-\frac{1}{2}}) \cap \mathcal{D}(B^{\frac{1}{2}})\right\}.
\end{aligned}$$

Lemma 9 *For all* $\omega_{\xi,\eta} \in C_2$ *one has* $\omega_{\xi,A^{-\frac{1}{2}}\eta}, \omega_{\xi,u^*B^{\frac{1}{2}}\eta} \in \mathcal{I}$. *Moreover,*

$$\xi_\Omega\left(\omega_{\xi,\eta}\right) = \xi\left(\omega_{\xi,A^{-\frac{1}{2}}\eta}\right), \quad \xi\left(\omega_{\xi,u^*B^{\frac{1}{2}}\eta}\right) = \lambda^{\frac{i}{4}} Ju^*J\xi\left(\omega_{\xi,A^{-\frac{1}{2}}\eta}\right).$$

The following set is a σ*-weak-weak core for* Λ_2*:*

$$\mathcal{D} = \left\{(x \otimes 1)(\omega_{\xi,\eta} \otimes \iota)(\tilde{W}_\Omega) \,|\, x \in C_1^0,\, \omega_{\xi,\eta} \in C_2\right\}.$$

Moreover, if $x \in C_1$ *and* $\omega_{\xi,\eta} \in C_2$*, then*

$$\Lambda_2((x \otimes 1)(\omega_{\xi,\eta} \otimes \iota)(\tilde{W}_\Omega)) = \Lambda_P(x) \otimes \xi(\omega_{\xi,A^{-\frac{1}{2}}\eta}).$$

Proof. Let $\omega_{\xi,\eta} \in \mathcal{I}_\Omega$ and $\eta \in \mathcal{D}(A^{-\frac{1}{2}})$. Let e_n be self-adjoint elements, like in Lemma 2, for the operator A. When $x \in \mathcal{N}_\varphi$, we have

$$\begin{aligned}
|\omega_{\xi,A^{-\frac{1}{2}}\eta}(e_n x^*)| &= |\langle e_n x^* \xi, A^{-\frac{1}{2}}\eta\rangle| = |\langle (A^{-\frac{1}{2}}e_n)x^*\xi, \eta\rangle| \\
&= |\langle (xA^{-\frac{1}{2}}e_n)^*\xi, \eta\rangle| \leq C||\Lambda_\Omega(xA^{-\frac{1}{2}}e_n)||,
\end{aligned}$$

because $xA^{-\frac{1}{2}}e_n A^{\frac{1}{2}}$ is bounded and its closure, which equals to xe_n, belons to $\mathcal{N}_\varphi$. Thus, we obtain

$$|\omega_{\xi,A^{-\frac{1}{2}}\eta}(e_n x^*)| \leq C||\Lambda(xe_n)|| = C||J\sigma_{-\frac{i}{2}}(e_n)J\Lambda(x)|| \to C||\Lambda(x)||.$$

Since $|\omega_{\xi,A^{-\frac{1}{2}}\eta}(e_n x^*)| \to |\omega_{\xi,A^{-\frac{1}{2}}\eta}(x^*)|$, we conclude that $\omega_{\xi,A^{-\frac{1}{2}}\eta}$ is in $\mathcal{I}$. Moreover, for all $x \in \mathcal{N}_\varphi$, we have

$$\begin{aligned}
\langle \xi_\Omega(\omega_{\xi,\eta}), J\sigma_{-\frac{i}{2}}(e_n)J\Lambda(x)\rangle &= \langle \xi_\Omega(\omega_{\xi,\eta}), \Lambda(xe_n)\rangle = \langle \xi_\Omega(\omega_{\xi,\eta}), \Lambda_\Omega(xA^{-\frac{1}{2}}e_n)\rangle \\
&= \omega_{\xi,\eta}(A^{-\frac{1}{2}}e_n x^*) = \langle e_n x^*\xi, A^{-\frac{1}{2}}\eta\rangle \\
&= \omega_{\xi,A^{-\frac{1}{2}}\eta}(e_n x^*) = \langle \xi(\omega_{\xi,A^{-\frac{1}{2}}\eta}), \Lambda(xe_n)\rangle \\
&= \langle \xi(\omega_{\xi,A^{-\frac{1}{2}}\eta}), J\sigma_{-\frac{i}{2}}(e_n)J\Lambda(x)\rangle.
\end{aligned}$$

Taking the limit when $n \to \infty$, we conclude that $\xi_\Omega(\omega_{\xi,\eta}) = \xi(\omega_{\xi,A^{-\frac{1}{2}}\eta})$.

Suppose that $\eta \in \mathcal{D}(B^{\frac{1}{2}})$. Let f_m be self-adjoint elements, like in Lemma 2, for the operator B. Note that f_m commute with e_n and u, also e_n commute with u. Let us show that $uB^{\frac{1}{2}}f_m A^{\frac{1}{2}}e_n$ is analytic w.r.t. σ. We have

$$\Psi(-(g-\gamma_t), g-\gamma_t) = \lambda^{-it^2}\Psi(-g,g)\Psi(g,\gamma_t)\Psi(\gamma_t,g),$$

so $\sigma_t(u) = \lambda^{-it^2}uA^{-it}B^{-it}$ and, using Lemma 2, we obtain

$$\begin{aligned}
\sigma_t(uB^{\frac{1}{2}}f_m A^{\frac{1}{2}}e_n) &= \lambda^{-it^2}uA^{-it}B^{-it}B^{\frac{1}{2}}\sigma_t(f_m)A^{\frac{1}{2}}\sigma_t(e_n) \\
&= \lambda^{-it^2}uB^{\frac{1}{2}-it}\sigma_t(f_m)A^{\frac{1}{2}-it}\sigma_t(e_n).
\end{aligned}$$

Define the following function from $\mathbb{C}$ to M:

$$f(z) = \lambda^{-iz^2}uB^{\frac{1}{2}-iz}\sigma_z(f_m)A^{\frac{1}{2}-iz}\sigma_z(e_n).$$

By Lemma 2, f is analytic, so $uB^{\frac{1}{2}}f_m A^{\frac{1}{2}}e_n$ is analytic, and we have

$$\sigma_{-\frac{i}{2}}(uB^{\frac{1}{2}}f_m A^{\frac{1}{2}}e_n) = \lambda^{\frac{i}{4}}u\sigma_{-\frac{i}{2}}(f_m)\sigma_{-\frac{i}{2}}(e_n).$$

Thus, for $x \in \mathcal{N}_\varphi$, $xu^*B^{\frac{1}{2}}f_m e_n A^{\frac{1}{2}}$ is bounded and its closure, which is equal to $xu^*B^{\frac{1}{2}}f_m A^{\frac{1}{2}}e_n$, belongs to $\mathcal{N}_\varphi$. Moreover,

$$\begin{aligned}
|\omega_{\xi,u^*B^{\frac{1}{2}}\eta}(e_n f_m x^*)| &= |\langle e_n f_m x^*\xi, u^*B^{\frac{1}{2}}\eta\rangle| = |\langle B^{\frac{1}{2}}f_m e_n u x^*\xi, \eta\rangle| \\
&= |\langle \left(xu^*B^{\frac{1}{2}}f_m e_n\right)^*\xi, \eta\rangle| \leq C||\Lambda(xu^*B^{\frac{1}{2}}f_m A^{\frac{1}{2}}e_n)|| \\
&\leq C||J\lambda^{\frac{i}{4}}u\sigma_{-\frac{i}{2}}(f_m)\sigma_{-\frac{i}{2}}(e_n)J\Lambda(x)||.
\end{aligned}$$

Taking the limit over m and n, we get

$$|\omega_{\xi,u^*B^{\frac{1}{2}}\eta}(x^*)| \leq C||JuJ\Lambda(x)|| \leq C||u||||\Lambda(x)||.$$

Thus, $\omega_{\xi,u^*B^{\frac{1}{2}}\eta} \in \mathcal{I}$. Moreover, for all $x \in \mathcal{N}_\varphi$, one has

$$\begin{aligned}
&\langle\xi(\omega_{\xi,u^*B^{\frac{1}{2}}\eta}), J\sigma_{-\frac{i}{2}}(e_n)\sigma_{-\frac{i}{2}}(f_m)J\Lambda(x)\rangle = \langle\xi(\omega_{\xi,u^*B^{\frac{1}{2}}\eta}), \Lambda(xe_nf_m)\rangle \\
&= \omega_{\xi,u^*B^{\frac{1}{2}}\eta}(e_nf_mx^*) = \langle e_nf_mx^*\xi, u^*B^{\frac{1}{2}}\eta\rangle = \langle x^*\xi, u^*B^{\frac{1}{2}}f_mA^{\frac{1}{2}}e_nA^{-\frac{1}{2}}\eta\rangle \\
&= \langle uB^{\frac{1}{2}}f_mA^{\frac{1}{2}}e_nx^*\xi, A^{-\frac{1}{2}}\eta\rangle = \omega_{\xi,A^{-\frac{1}{2}}\eta}\left((xu^*B^{\frac{1}{2}}f_mA^{\frac{1}{2}}e_n)^*\right) \\
&= \langle\xi(\omega_{\xi,A^{-\frac{1}{2}}\eta}), \Lambda(xu^*B^{\frac{1}{2}}f_mA^{\frac{1}{2}}e_n)\rangle = \langle\xi(\omega_{\xi,A^{-\frac{1}{2}}\eta}), J\lambda^{\frac{i}{4}}u\sigma_{-\frac{i}{2}}(f_m)\sigma_{-\frac{i}{2}}(e_n)J\Lambda(x)\rangle \\
&= \langle\lambda^{\frac{i}{4}}Ju^*J\xi(\omega_{\xi,A^{-\frac{1}{2}}\eta}), J\sigma_{-\frac{i}{2}}(e_n)\sigma_{-\frac{i}{2}}(f_m)J\Lambda(x)\rangle.
\end{aligned}$$

Taking the limit over m and n, we get $\xi(\omega_{\xi,u^*B^{\frac{1}{2}}\eta}) = \lambda^{\frac{i}{4}}Ju^*J\xi(\omega_{\xi,A^{-\frac{1}{2}}\eta})$. Now we want to prove that $\mathcal{D}$ is a σ-weak-weak core for Λ_2. Because $T = T_1 \otimes 1$, we know that $T^{\frac{1}{2}}(x \otimes 1)$ is bounded if and only if $T_1^{\frac{1}{2}}x$ is bounded. Thus, by Proposition 18, C_1^0 is a σ-strong*-norm core for Λ_P, and, by Proposition 17, it suffices to show that the set $\{(\omega \otimes \iota)(W_\Omega) \,|\, \omega \in C_2\}$ is a σ-strong*-norm core for $\hat{\Lambda}_\Omega$. Let $x = (\omega_{\xi,\eta} \otimes \iota)(W_\Omega)$ with $\omega_{\xi,\eta} \in \mathcal{I}_\Omega$. Let $L = \mathbb{N} \times \mathbb{N}$ with the product order and consider the net $x_{(n,m)} = (\omega_{\xi,e_nf_m\eta} \otimes \iota)(W_\Omega)$. Because $e_nf_m\eta \to \eta$, we have $x_{(n,m)} \to x$ in norm. Note that $e_nf_m\eta \in \mathcal{D}(A^{-\frac{1}{2}}) \cap \mathcal{D}(B^{\frac{1}{2}})$. Moreover, using the same techniques, one can show that $\omega_{\xi,e_nf_m\eta} \in \mathcal{I}_\Omega$. Thus, $\omega_{\xi,e_nf_m\eta} \in C_2$ and we have, for all $x \in M$ such that $xA^{\frac{1}{2}}$ is bounded and $\overline{xA^{\frac{1}{2}}} \in \mathcal{N}_\varphi$,

$$\begin{aligned}
\langle\hat{\Lambda}_\Omega(x_{(n,m)}), \Lambda_\Omega(x)\rangle &= \langle\xi_\Omega(\omega_{\xi,e_nf_m\eta}), \Lambda_\Omega(x)\rangle = \langle x^*\xi, e_nf_m\eta\rangle \\
&= \langle(xe_nf_m)^*\xi, \eta\rangle = \langle\xi_\Omega(\omega_{\xi,\eta}), \Lambda_\Omega(xe_nf_m)\rangle,
\end{aligned}$$

because $xe_nf_mA^{\frac{1}{2}}$ is bounded and $\overline{xe_nf_mA^{\frac{1}{2}}} = \overline{xA^{\frac{1}{2}}}e_nf_m \in \mathcal{N}_\varphi$, so

$$\begin{aligned}
\langle\hat{\Lambda}_\Omega(x_{(n,m)}), \Lambda_\Omega(x)\rangle &= \langle\xi_\Omega(\omega_{\xi,\eta}), J\sigma_{-\frac{i}{2}}(e_n)\sigma_{-\frac{i}{2}}(f_m)J\Lambda(\overline{xA^{\frac{1}{2}}})\rangle \\
&= \langle J\sigma_{\frac{i}{2}}(e_n)\sigma_{\frac{i}{2}}(f_m)J\xi_\Omega(\omega_{\xi,\eta}), \Lambda_\Omega(x)\rangle
\end{aligned}$$

Thus, $\hat{\Lambda}_\Omega(x_{(n,m)}) = J\sigma_{\frac{i}{2}}(e_n)\sigma_{\frac{i}{2}}(f_m)J\hat{\Lambda}_\Omega(x) \to \hat{\Lambda}_\Omega(x)$. ■

Next proposition describes the image by Λ_1 of typical elements from Λ_2. Let us define the unitaries

$$U = (\lambda_L^{\hat{G}^2} \otimes \alpha)(\Psi^\circ)^* \quad \text{and} \quad V = (\lambda_R^{\hat{G}^2} \otimes \alpha^{'})(\sigma\Psi^*).$$

Proposition 9 *Let $x \in C_1^0$ and $\omega \in M_*$ be such that $\omega u \in \mathcal{I}$, then $\Lambda_P \otimes \xi(\omega u) \in \mathcal{D}(T^{-\frac{1}{2}})$, $(x \otimes 1)(\omega \otimes \iota)(\tilde{W}_\Omega) \in \mathcal{N}_{\varphi_1}$ and*

$$\Lambda_1\left((x \otimes 1)(\omega \otimes \iota)(\tilde{W}_\Omega)\right) = UVT^{-\frac{1}{2}}\Lambda_P(x) \otimes \xi(\omega u).$$

First, we need some preliminary results.

Lemma 10 *Let J_1 be the modular conjugation associated with φ_1. Then, for all $F \in L^\infty(G)$,*
$$(\lambda_L^{\hat{G}^2} \otimes \alpha)(\Delta_G(F)) = J_1 \lambda_L(F)^* J_1.$$

Proof. Using Lemma 8 (3), we see that $((\gamma_1, \gamma_2) \mapsto L_{\gamma_1} R_{\gamma_2})$ is the standard implementation of the action α on $H = H_{\hat{\varphi}}$, so the operator J_1 is given by $J_1 \xi(\gamma_1, \gamma_2) = L_{-\gamma_1} R_{-\gamma_2} \hat{J} \xi(-\gamma_1, -\gamma_2)$, for $\xi \in L^2(\hat{G}^2, H)$. It is now easy to check the needed equality for $F = u_\gamma$ with $\gamma \in \hat{G}$. Because $(\lambda_L^{\hat{G}^2} \otimes \alpha) \circ \Delta_G$ and $J_1 \lambda_L(.)^* J_1$ are $*$-homomorphisms, this concludes the proof. ■

Define one parameter groups of automorphisms of $L^\infty(G)$: $\gamma_t(F)(x) = F(x - \gamma_t)$ and of M' : $\sigma'_t(x) = J \sigma_t(JxJ)J$. Note that $\sigma'_t \circ \alpha' = \alpha' \circ \gamma_t$. By analytic continuation, $\alpha'(F) \in \mathcal{D}(\sigma'_z)$ and $\sigma'_z(\alpha'(F)) = \alpha'(\gamma_z(F))\ \forall z \in \mathbb{C},\ F \in \mathcal{D}(\gamma_z)$.

Lemma 11 *Let $F \in L^\infty(G^2)$, $x \in \mathcal{N}_{\varphi_P}$, and $\omega \in \mathcal{I}$. If $F \in \mathcal{D}(\gamma_{-\frac{i}{2}} \otimes \iota)$, then $(\lambda_R^{\hat{G}^2} \otimes \alpha')(\sigma F) \in \mathcal{D}(\iota \otimes \sigma'_{-\frac{i}{2}})$, $(x \otimes 1)(\omega \otimes \iota)\left((\alpha \otimes \lambda_R)(F) \tilde{W} (\alpha \otimes \lambda_L)(F)\right) \in \mathcal{N}_{\varphi_1}$, and*
$$\begin{aligned}
&\Lambda_1 \left((x \otimes 1)(\omega \otimes \iota)\left((\alpha \otimes \lambda_R)(F) \tilde{W} (\alpha \otimes \lambda_L)(F)\right)\right) \\
&= (\lambda_L^{\hat{G}^2} \otimes \alpha)(F^\circ)(\iota \otimes \sigma'_{-\frac{i}{2}})\left((\lambda_R^{\hat{G}^2} \otimes \alpha')(\sigma F)\right) \Lambda_P(x) \otimes \xi(\omega).
\end{aligned}$$

Proof. Because $\mathcal{D}(\gamma_{-\frac{i}{2}}) \odot L^\infty(G)$ is a σ-strong* core for $\gamma_{-\frac{i}{2}} \otimes \iota$ and Λ_1 is σ-weak-weak closed, we can take $F \in \mathcal{D}(\gamma_{-\frac{i}{2}}) \odot L^\infty(G)$. By linearity, we can take $F = F_1 \otimes F_2$ with $F_1 \in \mathcal{D}(\gamma_{-\frac{i}{2}})$ and $F_2 \in L^\infty(G)$. If $x \in \mathcal{N}_{\varphi_P}$ and $\omega \in \mathcal{I}$, then
$$\begin{aligned}
&(x \otimes 1)(\omega \otimes \iota)\left((\alpha \otimes \lambda_R)(F) \tilde{W} (\alpha \otimes \lambda_L)(F)\right) \\
&= \lambda_R(F_2)(x \otimes 1)(\alpha(F_1).\omega.\alpha(F_1) \otimes \iota)(\tilde{W}) \lambda_L(F_2).
\end{aligned}$$

Because $F_1 \in \mathcal{D}(\gamma_{-\frac{i}{2}})$, we have $\alpha(F_1) \in \mathcal{D}(\sigma_{-\frac{i}{2}})$. Lemma 3 and the definition of Λ_1 imply $(x \otimes 1)(\alpha(F_1) \cdot \omega \cdot \alpha(F_1) \otimes \iota)(\tilde{W}) \in \mathcal{N}_{\varphi_1}$ and
$$\begin{aligned}
&\Lambda_1 \left((x \otimes 1)(\alpha(F_1) \cdot \omega \cdot \alpha(F_1) \otimes \iota)(\tilde{W})\right) \\
&= (1 \otimes \alpha(F_1))(1 \otimes J\sigma_{-\frac{i}{2}}(\alpha(F_1)^* J)(\Lambda_P(x) \otimes \xi(\omega)) \\
&= (1 \otimes \alpha(F_1))(1 \otimes \alpha'(\gamma_{-\frac{i}{2}}(F_1)))(\Lambda_P(x) \otimes \xi(\omega)).
\end{aligned}$$

Moreover, (14) gives $\lambda_L(F_2) \in N^{\varphi_1}$, so
$$\begin{aligned}
&\lambda_R(F_2)(x \otimes 1)(\alpha(F_1).\omega.\alpha(F_1) \otimes \iota)(\tilde{W}) \lambda_L(F_2) \in \mathcal{N}_{\varphi_1} \quad \text{and,} \\
&\Lambda_1 \left(\lambda_R(F_2)(x \otimes 1)(\alpha(F_1).\omega.\alpha(F_1) \otimes \iota)(\tilde{W}) \lambda_L(F_2)\right) \\
&= J_1 \lambda_L(F_2)^* J_1 \lambda_R(F_2)(1 \otimes \alpha(F_1))(1 \otimes \alpha'(\gamma_{-\frac{i}{2}}(F_1)))(\Lambda_P(x) \otimes \xi(\omega)) \\
&= J_1 \lambda_L(F_2)^* J_1 (1 \otimes \alpha(F_1)) \left((\lambda_R^{\hat{G}^2}(F_2) \otimes 1)(1 \otimes \alpha'(\gamma_{-\frac{i}{2}}(F_1)))\right) (\Lambda_P(x) \otimes \xi(\omega)) \\
&\quad \text{(because } \lambda_R(F_2) = \lambda_R^{\hat{G}^2}(F_2) \otimes 1 \text{ commute with } 1 \otimes \alpha(F_1)) \\
&= (\lambda_L^{\hat{G}^2} \otimes \alpha)(\Delta_G(F_2) 1 \otimes F_1)(\lambda_R^{\hat{G}^2} \otimes \alpha')((F_2 \otimes 1)(1 \otimes \gamma_{-\frac{i}{2}}(F_1)))(\Lambda_P(x) \otimes \xi(\omega)),
\end{aligned}$$

where we used Lemma 10 in the last equality. Note that

$$(\Delta_G(F_2)1\otimes F_1)(g,h) = F_2(g+h)F_1(h) = F(h,g+h) = F^\circ(g,h),$$

and because

$$\begin{aligned}(\lambda_R^{\hat{G}^2}\otimes\alpha^{'})((F_2\otimes 1)(1\otimes\gamma_{-\frac{i}{2}}(F_1))) &= (\lambda_R^{\hat{G}^2}\otimes\alpha^{'})((\iota\otimes\gamma_{-\frac{i}{2}})(\sigma F))\\ &= (\iota\otimes\sigma^{'}_{-\frac{i}{2}})((\lambda_R^{\hat{G}^2}\otimes\alpha^{'})(\sigma F)),\end{aligned}$$

we conclude the proof. ■

Lemma 12 *The operator $(\iota\otimes\sigma^{'}_{-\frac{i}{2}})(V)$ is normal, affilated with $\mathcal{L}(\widehat{G}^2)\otimes M^{'}$, and its polar decomposition is $(\iota\otimes\sigma^{'}_{-\frac{i}{2}})(V) = V(T_1^{-\frac{1}{2}}\otimes 1) = VT^{-\frac{1}{2}}$.*

Proof. We have $(\iota\otimes\gamma_t)(\sigma\Psi^*)(g,h) = \Psi^*(h,g)\Psi^*(-\gamma_t,g)$, so

$$(\iota\otimes\sigma^{'}_t)(V) = (\lambda_R^{\hat{G}^2}\otimes\alpha^{'})((\iota\otimes\gamma_t)(\sigma\Psi^*)) = V(T_1^{-it}\otimes 1).$$

We conclude the proof by applying Proposition 1. ■

Proof of Proposition 9. Let $x\in\mathcal{C}_1^0$ and $\omega\in M_*$ such that $\omega\cdot u\in\mathcal{I}$. By Lemma 12, $\Lambda_P(x)\otimes\xi(\omega\cdot u)\in\mathcal{D}((\iota\otimes\sigma^{'}_{-\frac{i}{2}})(V))$ and

$$(\iota\otimes\sigma^{'}_{-\frac{i}{2}})(V)\Lambda_P(x)\otimes\xi(\omega\cdot u) = VT^{-\frac{1}{2}}\Lambda_P(x)\otimes\xi(\omega\cdot u).$$

By Lemma 1, $V(n)\to V$ σ-strongly* and

$$(\iota\otimes\sigma^{'}_{-\frac{i}{2}})(V(n))\Lambda_P(x)\otimes\xi(\omega\cdot u)\to VT^{-\frac{1}{2}}\Lambda_P(x)\otimes\xi(\omega\cdot u),$$

where

$$V(n) = \sqrt{\frac{n}{\pi}}\int e^{-nt^2}(\iota\otimes\sigma^{'}_t)(V)dt = (\lambda_R^{\hat{G}^2}\otimes\alpha^{'})(\sigma\Psi^*(n)),$$

with $\Psi^*(n) = \sqrt{\frac{n}{\pi}}\int e^{-nt^2}(\gamma_t\otimes\iota)(\Psi^*)dt$. So $\Psi^*(n)$ is analytic w.r.t. $(t\mapsto\gamma_t\otimes\iota)$ and $\Psi^*(n)\to\Psi^*$ σ-strongly*. Now we can apply Lemma 11 to $\Psi^*(n)$ and $\omega\cdot u$: $(x\otimes 1)(\omega\cdot u\otimes\iota)\Big((\alpha\otimes\lambda_R)(\Psi^*(n))\tilde{W}(\alpha\otimes\lambda_L)(\Psi^*(n))\Big)\in\mathcal{N}_{\varphi_1}$ and

$$\begin{aligned}&\Lambda_1\Big((x\otimes 1)(\omega\cdot u\otimes\iota)\Big((\alpha\otimes\lambda_R)(\Psi^*(n))\tilde{W}(\alpha\otimes\lambda_L)(\Psi^*(n))\Big)\Big)\\ &\quad = (\lambda_L^{\hat{G}^2}\otimes\alpha)(\Psi^*(n)^\circ)(\iota\otimes\sigma^{'}_{-\frac{i}{2}})(V(n))\Lambda_P(x)\otimes\xi(\omega\cdot u).\end{aligned}$$

Note that

$(\alpha\otimes\lambda_R)(\Psi^*(n))\tilde{W}(\alpha\otimes\lambda_L)(\Psi^*(n))\to(\alpha\otimes\lambda_R)(\Psi^*)\tilde{W}(\alpha\otimes\lambda_L)(\Psi^*)$ $\sigma-$weakly, so,

$(x\otimes 1)(\omega\cdot u\otimes\iota)\Big((\alpha\otimes\lambda_R)(\Psi^*(n))\tilde{W}(\alpha\otimes\lambda_L)(\Psi^*(n))\Big)\to(x\otimes 1)(\omega\otimes\iota)(\tilde{W}_\Omega)$ $\sigma-$weakly,

and

$(\lambda_L^{\hat{G}^2}\otimes\alpha)(\Psi^*(n)^\circ)(\iota\otimes\sigma^{'}_{-\frac{i}{2}})(V(n))\Lambda_P(x)\otimes\xi(\omega.u)\to UVT^{-\frac{1}{2}}\Lambda_P(x)\otimes\xi(\omega.u)$ weakly.

Because Λ_1 is σ-weak-weak closed, this concludes the proof.

Lemma 13 *Let $\eta \in \mathcal{D}(B^{\frac{1}{2}})$, $\xi \in H$ and $x \in \mathcal{C}_1$. Then*

$(x\otimes 1)(\omega_{\xi,\eta}\otimes\iota)(\tilde{W}_\Omega)T^{\frac{1}{2}}$ *is bounded and its closure is* $\overline{T^{\frac{1}{2}}(x\otimes 1)}(\omega_{\xi,B^{\frac{1}{2}}\eta}\otimes\iota)(\tilde{W}_\Omega)$.

Proof. Using (15), for all $t \in \mathbb{R}$, we have $\tilde{W}_\Omega(1\otimes T^{it})\tilde{W}_\Omega^* = B^{it}\otimes T^{it}$, so $\tilde{W}_\Omega(1\otimes T^{\frac{1}{2}})\tilde{W}_\Omega^* = B^{\frac{1}{2}}\otimes T^{\frac{1}{2}}$. Let $\eta \in \mathcal{D}(B^{\frac{1}{2}})$, $\xi \in H$, $x \in \mathcal{C}_1$, $f \in \mathcal{D}(T^{\frac{1}{2}})$, and $l \in L^2(\hat{G}^2, H)$, then

$$\begin{aligned}
\langle (x\otimes 1)(\omega_{\xi,\eta}\otimes\iota)(\tilde{W}_\Omega)T^{\frac{1}{2}}f, l\rangle &= \langle \tilde{W}_\Omega\xi\otimes T^{\frac{1}{2}}f, \eta\otimes(x\otimes 1)^*l\rangle \\
&= \langle (B^{\frac{1}{2}}\otimes T^{\frac{1}{2}})\tilde{W}_\Omega\xi\otimes f, \eta\otimes(x\otimes 1)^*l\rangle \\
&= \langle (1\otimes\overline{(x\otimes 1)T^{\frac{1}{2}}})\tilde{W}_\Omega\xi\otimes f, B^{\frac{1}{2}}\eta\otimes l\rangle \\
&= \langle \overline{T^{\frac{1}{2}}(x\otimes 1)}(\omega_{\xi,B^{\frac{1}{2}}\eta}\otimes\iota)(\tilde{W}_\Omega)f, l\rangle.
\end{aligned}$$

Thus, we have $(x\otimes 1)(\omega_{\xi,\eta}\otimes\iota)(\tilde{W}_\Omega)T^{\frac{1}{2}} \subset \overline{T^{\frac{1}{2}}(x\otimes 1)}(\omega_{\xi,B^{\frac{1}{2}}\eta}\otimes\iota)(\tilde{W}_\Omega)$. Because $\mathcal{D}(T^{\frac{1}{2}})$ is dense, this concludes the proof. ∎

Proposition 10 *Let $x \in \mathcal{C}_1^0$ and $\omega_{\xi,\eta} \in C_2$. Then*

$$(x\otimes 1)(\omega_{\xi,\eta}\otimes\iota)(\tilde{W}_\Omega) \in \mathcal{N}_\mu \cap \mathcal{N}_{\varphi_2} \quad \textit{and}$$

$$\Lambda_\mu\left((x\otimes 1)(\omega_{\xi,\eta}\otimes\iota)(\tilde{W}_\Omega)\right) = \lambda^{\frac{i}{4}}UV(1\otimes Ju^*J)\Lambda_2\left((x\otimes 1)(\omega_{\xi,\eta}\otimes\iota)(\tilde{W}_\Omega)\right).$$

Proof. Let $x \in \mathcal{C}_1^0$ and $\omega_{\xi,\eta} \in C_2$. By Lemma 13, $(x\otimes 1)(\omega_{\xi,\eta}\otimes\iota)(\tilde{W}_\Omega)T^{\frac{1}{2}}$ is bounded and its closure is $\overline{T^{\frac{1}{2}}(x\otimes 1)}(\omega_{\xi,B^{\frac{1}{2}}\eta}\otimes\iota)(\tilde{W}_\Omega)$. Moreover, by Lemma 9, $\omega_{\xi,B^{\frac{1}{2}}\eta}\cdot u = \omega_{\xi,u^*B^{\frac{1}{2}}\eta} \in \mathcal{I}$, so we can apply Proposition 9, and we find that $(x\otimes 1)(\omega_{\xi,B^{\frac{1}{2}}\eta}\otimes\iota)(\tilde{W}_\Omega) \in \mathcal{N}_{\varphi_1}$ and

$$\Lambda_1\left((x\otimes 1)(\omega_{\xi,B^{\frac{1}{2}}\eta}\otimes\iota)(\tilde{W}_\Omega)\right) = UVT^{-\frac{1}{2}}\Lambda_P(x)\otimes\xi(\omega_{\xi,u^*B^{\frac{1}{2}}\eta}).$$

Finally, using Proposition 18, and because $T^{\frac{1}{2}}$ commutes with UV, we find that $\overline{T^{\frac{1}{2}}(x\otimes 1)}(\omega_{\xi,B^{\frac{1}{2}}\eta}\otimes\iota)(\tilde{W}_\Omega) \in \mathcal{N}_{\varphi_1}$ and

$$\Lambda_1\left(\overline{T^{\frac{1}{2}}(x\otimes 1)}(\omega_{\xi,B^{\frac{1}{2}}\eta}\otimes\iota)(\tilde{W}_\Omega)\right) = UV\Lambda_P(x)\otimes\xi(\omega_{\xi,u^*B^{\frac{1}{2}}\eta}).$$

By Lemma 9, $\xi(\omega_{\xi,u^*B^{\frac{1}{2}}\eta}) = \lambda^{\frac{i}{4}}Ju^*J\xi_\Omega(\omega_{\xi,\eta})$, so

$$\begin{aligned}
&\Lambda_1\left(\overline{T^{\frac{1}{2}}(x\otimes 1)}(\omega_{\xi,B^{\frac{1}{2}}\eta}\otimes\iota)(\tilde{W}_\Omega)\right) = \lambda^{\frac{i}{4}}UV(1\otimes Ju^*J)\Lambda_P(x)\otimes\xi_\Omega(\omega_{\xi,\eta}) \\
&= \lambda^{\frac{i}{4}}UV(1\otimes Ju^*J)\Lambda_2\left((x\otimes 1)(\omega_{\xi,\eta}\otimes\iota)(\tilde{W}_\Omega)\right).
\end{aligned}$$

∎ *Proof of Theorem 6.* Let $\mathcal{D}$ be the σ-weak-weak core for Λ_2 introduced before Lemma 9. By Proposition 10, $\mathcal{D} \subset \mathcal{N}_\mu \cap \mathcal{N}_{\varphi_2}$ and there is a unitary Z such that $\Lambda_2(x) = Z\Lambda_\mu(x)$, for all $x \in \mathcal{D}$. By Proposition 19, $\varphi_2 = \tilde{\mu}_\Omega$, so $\hat{\varphi}_\Omega = \mu_\Omega$.

5 Examples

5.1 Twisting of the $az + b$ group

Our aim is to prove Theorem 3. According to Section 2.7, if H is a closed abelian subgroup of a l.c. group G, then $H < (\mathcal{L}(G), \hat{\Delta}_G)$ is an abelian stable co-subgroup. The morphism $\alpha : L^\infty(\widehat{H}) \to \mathcal{L}(G)$ is given by $\alpha(u_h) = \lambda_G(h)$, and the morphism $(t \mapsto \gamma_t) : \mathbb{R} \to \widehat{H}$ by $\langle \gamma_t, h\rangle = \delta_G^{-it}(h)$. Let $G = \mathbb{C}^* \ltimes \mathbb{C}$ and $K \subset G$ be the subgroup $K = \{(z, 0),\ z \in \mathbb{C}^*\}$. The modular function of G is $\delta_G(z, w) = |z|^{-1}$, for all $z \in \mathbb{C}^*$, $\omega \in \mathbb{C}$, and $\langle \gamma_t, z\rangle = |z|^{it}$, for all $z \in \mathbb{C}^*, t \in \mathbb{R}$. Let us identify $\widehat{\mathbb{C}^*}$ with $\mathbb{Z} \times \mathbb{R}^*_+$:

$$\mathbb{Z} \times \mathbb{R}^*_+ \to \widehat{\mathbb{C}^*}, \quad (n, \rho) \mapsto \gamma_{n,\rho} = (re^{i\theta} \mapsto e^{i \ln r \ln \rho} e^{in\theta}).$$

Then $\gamma_t = (0, e^t) \in \mathbb{Z} \times \mathbb{R}^*_+$. For any $x \in \mathbb{R}$, there is a bicharacter on $\mathbb{Z} \times \mathbb{R}^*_+$: $\Psi_x((n, \rho), (k, r)) = e^{ix(k \ln \rho - n \ln r)}$. Let (M_x, Δ_x) be the l.c. quantum group obtained by twisting. Then $\Psi_x((n, \rho), \gamma_t^{-1}) = e^{ixtn} = u_{e^{ixt}}((n, \rho))$, and we get the operator A_x deforming the Plancherel weight φ:

$$A_x^{it} = \alpha(u_{e^{2ixt}}) = \lambda^G_{(e^{itx},0)}.$$

Since $\Psi_x(\gamma_t, \gamma_s) = 1$, for all $s, t \in \mathbb{R}$, the twisted left-invariant weight φ_x satisfies $[D\varphi_x : D\varphi]_t = A_x^{it} = \lambda^G_{(e^{itx},0)}$. The modular element of the twisted quantum group is

$$\delta_x^{it} = \alpha(\Psi_x(\cdot, \gamma_t)\Psi_x(-\gamma_t, \cdot)) = \lambda^G_{(e^{-2itx},0)},$$

so δ_x is not affiliated with the center of $\mathcal{L}(G)$, and the twisted quantum group is not a Kac algebra. Let us look if (M_x, Δ_x) is isomorphic for different values of x. Since Ψ_x is antisymmetric, $\Psi_{-x} = \Psi_x^*$, and Δ is cocommutative, we have $\Delta_{-x} = \sigma\Delta_x$. Thus, $(M_{-x}, \Delta_{-x}) \simeq (M_x, \Delta_x)^{\mathrm{op}}$. Moreover, using the Fourier transformation in the first variable, one has immediately $\mathrm{Sp}(\delta_x) = q_x^{\mathbb{Z}} \cup \{0\}$, where $q_x = e^{-2x}$. Thus, if $x \neq y$, $x > 0, y > 0$, one has $q_x^{\mathbb{Z}} \neq q_y^{\mathbb{Z}}$ and, consequently, (M_x, Δ_x) and (M_y, Δ_y) are not isomorphic.

In order to finish the proof of Theorem 3, we must compute the dual l.c. quantum group. The action of K^2 on $L^\infty(G)$ can be lifted to its Lie algebra $\mathbb{C}^2$ which does not change the result of deformation (see [6]) but simplifies calculations. The group $\mathbb{C}$ is self-dual with the duality $(z_1, z_2) \mapsto \exp(i\mathrm{Im}(z_1 z_2))$. Let $x \in \mathbb{R}$. The lifted bicharacter on $\mathbb{C}$ is $\Psi_x(z_1, z_2) = \exp(ix\mathrm{Im}(z_1\overline{z}_2))$. The action ρ of $\mathbb{C}^2$ on $L^\infty(G)$ is

$$\rho_{z_1,z_2}(f)(w_1, w_2) = f(e^{z_2 - z_1} w_1, e^{-z_1} w_2). \tag{16}$$

Let $N = \mathbb{C}^2 \ltimes L^\infty(G)$ and θ be the dual action of $\mathbb{C}^2$ on N. One has, for all $z, w \in \mathbb{C}$, $\Psi_x(w, z) = u_{x\overline{z}}(w)$. So, the twisted dual action is

$$\theta^{\Psi_x}_{z_1,z_2} = \lambda_{-x\overline{z}_1, x\overline{z}_2}\theta_{z_1,z_2}(\cdot)\lambda^*_{-x\overline{z}_1, x\overline{z}_2}. \tag{17}$$

Let $\widehat{M_x}$ be the fixed point algebra. We will construct two operators affiliated with $\widehat{M_x}$ which generate $\widehat{M_x}$. Let a and b be the coordinate functions on G, and

$\alpha = \pi(a)$, $\beta = \pi(b)$. Then α and β are normal operators affiliated with N, and (16) gives

$$\lambda_{z_1,z_2}\alpha\lambda^*_{z_1,z_2} = e^{z_2-z_1}\alpha, \quad \lambda_{z_1,z_2}\beta\lambda^*_{z_1,z_2} = e^{-z_1}\beta. \tag{18}$$

Now, using (17) and (18), we find

$$\theta^{\Psi_x}_{z_1,z_2}(\alpha) = e^{x(\overline{z}_1+\overline{z}_2)}\alpha\,, \quad \theta^{\Psi_x}_{z_1,z_2}(\beta) = e^{x\overline{z}_1}\beta. \tag{19}$$

Let T_l and T_r be the infinitesimal generators of the left and right translations, so T_l and T_r are affiliated with N and $\lambda_{z_1,z_2} = \exp\left(i\mathrm{Im}(z_1T_l)\right)\exp\left(i\mathrm{Im}(z_2T_r)\right)$. Then $\lambda(f) = f(T_l, T_r)$, for all $f \in L^\infty(\mathbb{C}^2)$.

Lemma 14 *Let $L = e^{xT_l^*}$ and $R = e^{xT_r^*}$, then*

- *(β, L) is a e^x-commuting pair.*
- *(β, R) is a 1-commuting pair.*
- *(α, R) is a e^{-x}-commuting pair.*
- *(α, L) is a e^x-commuting pair.*

Proof. Note that $\mathrm{Ph}(L) = e^{-ix\mathrm{Im}T_l} = \lambda_{-x,0}$ and $|L|^{is} = e^{isx\mathrm{Re}T_l} = \lambda_{isx,0}$, so (18) gives $|L|^{is}\beta|L|^{-is} = e^{-isx}\beta$ and $\mathrm{Ph}(L)\beta\mathrm{Ph}(L)^* = e^x\beta$ which means that (β, L) is a e^x-commuting pair. The proof of the other assertions is similar. ∎

Define $U = \lambda(\Psi_x)$ and $\hat{\alpha} = U^*\alpha U$, $\hat{v} = \mathrm{Ph}(L)\mathrm{Ph}(\beta)$ and $\hat{B} = |L||\beta|$. Then $\hat{\alpha}$ is normal, $\hat{B}$ is positive self adjoint, both affiliated with N, and $\hat{v} \in N$ is unitary.

Proposition 11 *$\hat{\alpha}$ and $\hat{B}$ are affiliated with $\widehat{M}_x$ and $\hat{v} \in \widehat{M}_x$. Moreover,*

$$\left\{f(\hat{\alpha})g(\hat{B})h(\hat{v}),\ f \in L^\infty(\mathbb{C}), g \in L^\infty(\mathbb{R}^*_+), h \in L^\infty(\mathbb{S}^1)\right\}'' = \widehat{M}_x.$$

Proof. We have

$$\begin{aligned}
\theta^{\Psi_x}_{z_1,z_2}(U) &= \lambda\left(\Psi_x(.-z_1, .-z_2)\right) \\
&= Ue^{ix\mathrm{Im}(-\overline{z}_2T_l)}e^{ix\mathrm{Im}(\overline{z}_1T_r)}\Psi_x(z_1,z_2) \\
&= U\lambda_{-x\overline{z}_2,x\overline{z}_1}\Psi_x(z_1,z_2).
\end{aligned}$$

This implies, using (19) and (18):

$$\theta^{\Psi_x}_{z_1,z_2}(\hat{\alpha}) = e^{x(\bar{z}_1+\bar{z}_2)}U^*\lambda_{x\bar{z}_2,-x\bar{z}_1}\alpha\lambda^*_{x\bar{z}_2,-x\bar{z}_1}U = \hat{\alpha}.$$

Also,

$$\theta^{\Psi_x}_{z_1,z_2}(\hat{B}) = e^{x\mathrm{Re}(T_l-\bar{z}_1)}e^{x\mathrm{Re}(\bar{z}_1)}|\beta| = \hat{B}$$

and

$$\theta^{\Psi_x}_{z_1,z_2}(\hat{v}) = e^{ix\mathrm{Im}(T_l^*-\bar{z}_1)}e^{ix\mathrm{Im}(\bar{z}_1)}\mathrm{Ph}(\beta) = \hat{v}.$$

Thus, $\hat{\alpha}$ and $\hat{B}$ are affiliated with $\widehat{M_x}$ and $\hat{v} \in \widehat{M_x}$. Let

$$\mathcal{W} = \left\{ zf(\hat{\alpha})g(\hat{B})h(\hat{v})y,\ f \in L^\infty(\mathbb{C}), g \in L^\infty(\mathbb{R}_+^*), h \in L^\infty(\mathbb{S}^1),\ z, y \in \lambda(L^\infty(\mathbb{C}^2)) \right\}''.$$

By Lemma 15, it suffices to show that $\mathcal{W} = N$. Note that

$$\left\{ zf(\hat{\alpha}),\ f \in L^\infty(\mathbb{C}), z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' = \left\{ zU^* f(\alpha)U,\ f \in L^\infty(\mathbb{C}), z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}''.$$

Substituting $z \mapsto zU$, we get

$$\left\{ zf(\hat{\alpha}),\ f \in L^\infty(\mathbb{C}), z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' = \left\{ zf(\alpha)U,\ f \in L^\infty(\mathbb{C}), z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}''.$$

Observe that

$$\left\{ f(\alpha)z\,,\ f \in L^\infty(\mathbb{C}),\ z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' = \left\{ zf(\alpha)\,,\ f \in L^\infty(\mathbb{C}),\ z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'',$$

so

$$\left\{ zf(\hat{\alpha}),\ f \in L^\infty(\mathbb{C}), z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' = \left\{ f(\alpha)zU,\ f \in L^\infty(\mathbb{C}), z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}''.$$

Substituting $z \mapsto zU^*$, we get

$$\left\{ zf(\hat{\alpha}),\ f \in L^\infty(\mathbb{C}), z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' = \left\{ f(\alpha)z,\ f \in L^\infty(\mathbb{C}), z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'',$$

so

$$\mathcal{W} = \left\{ f(\alpha)zg(\hat{B})h(\hat{v})y,\ f \in L^\infty(\mathbb{C}), g \in L^\infty(\mathbb{R}_+^*), h \in L^\infty(\mathbb{S}^1),\ z, y \in \lambda(L^\infty(\mathbb{C}^2)) \right\}''.$$

Note that

$$\begin{aligned} &\left\{ zg(\hat{B}),\ g \in L^\infty(\mathbb{R}_+^*),\ z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' = \left\{ z\hat{B}^{is},\ s \in \mathbb{R},\ z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' \\ &\quad = \left\{ ze^{ist\mathrm{Re}T_l}|\beta|^{is},\ s \in \mathbb{R},\ x \in \lambda(L^\infty(\mathbb{C}^2)) \right\}''. \end{aligned}$$

Substitution $z \mapsto ze^{-ist\mathrm{Re}T_l}$ gives

$$\begin{aligned} &\left\{ zg(\hat{B}),\ g \in L^\infty(\mathbb{R}_+^*),\ z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' = \left\{ z|\beta|^{is},\ s \in \mathbb{R},\ z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' \\ &\quad = \left\{ zg(|\beta|),\ g \in L^\infty(\mathbb{R}_+^*),\ z \in \lambda(L^\infty(\mathbb{C}^2)) \right\}''. \end{aligned}$$

Also, one can prove that

$$\left\{ h(\hat{v})y,\ h \in L^\infty(\mathbb{S}^1),\ y \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' = \left\{ h(\mathrm{Ph}\beta)y,\ h \in L^\infty(\mathbb{S}^1),\ y \in \lambda(L^\infty(\mathbb{C}^2)) \right\}''.$$

Thus,

$$\begin{aligned} \mathcal{W} &= \left\{ f(\alpha)zg(|\beta|)h(\mathrm{Ph}\beta)y,\ f \in L^\infty(\mathbb{C}), g \in L^\infty(\mathbb{R}_+^*), h \in L^\infty(\mathbb{S}^1),\ z, y \in \lambda(L^\infty(\mathbb{C}^2)) \right\}'' \\ &= \left\{ f(\alpha)zg(\beta)y,\ f, g \in L^\infty(\mathbb{C}),\ z, y \in \lambda(L^\infty(\mathbb{C}^2)) \right\}''. \end{aligned}$$

Commuting back $f(\alpha)$ and z, we have the result. ■

Let $\hat{\beta} = \hat{v}\hat{B}$. Then $\hat{\beta}$ is a closed (non normal) operator affiliated with $\widehat{M_x}$. Let us give now the commutation relations between $\hat{\alpha}$, $\hat{\beta}$.

Proposition 12 *α and $T_l^*+T_r^*$ strongly commute, and $\hat\alpha = e^{x(T_l^*+T_r^*)}$, so the polar decomposition of $\hat\alpha$ is*

$$Ph(\hat\alpha)=e^{-ixIm(T_l+T_r)}Ph(\alpha)=Ph(L)Ph(R)Ph(\alpha),\ |\hat\alpha|=e^{xRe(T_l+T_r)}|\alpha|=|L||R||\alpha|.$$

Moreover, the following relations hold with $q=e^{2x}$:

- $\hat\beta\hat\beta^*=q\hat\beta^*\hat\beta$,
- $(\hat\alpha,\hat\beta)$ *is a $\sqrt{q}$-commuting pair.*

Proof. Since

$$e^{i\mathrm{Im}(z(T_l^*+T_r^*))}\alpha e^{-i\mathrm{Im}(z(T_l^*+T_r^*))}=\lambda_{-\overline{z},-\overline{z}}\alpha\lambda^*_{-\overline{z},-\overline{z}}=e^{-\overline{z}+\overline{z}}\alpha=\alpha,$$

$T_l^*+T_r^*$ and α strongly commute. Moreover, since $e^{ix\mathrm{Im}T_lT_l^*}=1$,

$$\hat\alpha=e^{-ix\mathrm{Im}T_lT_r^*}\alpha e^{ix\mathrm{Im}T_lT_r^*}=e^{-ix\mathrm{Im}T_l(T_l+T_r)^*}\alpha e^{ix\mathrm{Im}T_l(T_l+T_r)^*}.$$

This equality, the strong commutativity of $T_l^*+T_r^*$ with α, and the equality $e^{-ix\mathrm{Im}T_l\omega}\alpha e^{ix\mathrm{Im}T_l\omega}=e^{x\omega}\alpha$ imply $\hat\alpha=e^{x(T_l^*+T_r^*)}$. The polar decomposition of $\hat\alpha$ follows. All the relations can be checked using Lemma 14. ■

We shall give now a nice formula for $\hat\Delta_x$. Let us define the following (closed non-normal) operator affiliated with $\widehat{M}_x\otimes\widehat{M}_x$: $\hat\Delta_x(\hat\beta)=\hat\Delta_x(\hat v)\hat\Delta_x(\hat B)$.

Proposition 13

$$\hat\Delta_x(\hat\alpha)=\hat\alpha\otimes\hat\alpha\quad and\quad \hat\Delta_x(\hat\beta)=\hat\alpha\otimes\hat\beta\dot{+}\hat\beta\otimes 1.$$

Proof. Proposition 7 gives $\hat\Delta_x=\Upsilon\Gamma(\cdot)\Upsilon^*$, where $\Upsilon=e^{ix\mathrm{Im}T_r\otimes T_l^*}$, and Γ is uniquely characterized by two properties:

- $\Gamma(T_l)=T_l\otimes 1$, $\Gamma(T_r)=1\otimes T_r$;
- Γ restricted to $L^\infty(G)$ coincides with the comultiplication Δ_G.

With $V=\Upsilon\Gamma(U^*)$, we have $\hat\Delta_x(\hat\alpha)=V(\alpha\otimes\alpha)V^*$, so it suffices to show that $(U\otimes U)V$ commutes with $\alpha\otimes\alpha$. Indeed in this case

$$\hat\Delta_x(\hat\alpha)=V(\alpha\otimes\alpha)V^*=(U^*\otimes U^*)(U\otimes U)V(\alpha\otimes\alpha)V^*(U^*\otimes U^*)(U\otimes U)=\hat\alpha\otimes\hat\alpha.$$

Let us show that $(U\otimes U)V$ commutes with $\alpha\otimes\alpha$. From $U=e^{ix\mathrm{Im}T_lT_r^*}$ one has

$$\Gamma(U^*)=e^{-ix\mathrm{Im}T_l\otimes T_r^*},\quad U\otimes U=e^{ix\mathrm{Im}(T_lT_r^*\otimes 1+1\otimes T_lT_r^*)},$$

so $V=e^{-ix\mathrm{Im}(T_r^*\otimes T_l+T_l\otimes T_r^*)}$ and

$$(U\otimes U)V=e^{ix\mathrm{Im}(T_lT_r^*\otimes 1+1\otimes T_lT_r^*-T_r^*\otimes T_l-T_l\otimes T_r^*)}.$$

Remark that

$$T_lT_r^*\otimes 1+1\otimes T_lT_r^*-T_r^*\otimes T_l-T_l\otimes T_r^*=(T_l\otimes 1-1\otimes T_l)(T_r^*\otimes 1-1\otimes T_r^*),$$

so it is enough to show that $T_l \otimes 1 - 1 \otimes T_l$ and $T_r^* \otimes 1 - 1 \otimes T_r^*$ strongly commute with $\alpha \otimes \alpha$, which follows from

$$\begin{array}{rcl} e^{i\mathrm{Im}z(T_r^*\otimes 1-1\otimes T_r^*)}(\alpha\otimes\alpha)e^{-i\mathrm{Im}z(T_r^*\otimes 1-1\otimes T_r^*)} & = & (\lambda_{0,-\overline{z}}\otimes\lambda_{0,\overline{z}})(\alpha\otimes\alpha)(\lambda_{0,-\overline{z}}\otimes\lambda_{0,\overline{z}})^* \\ & = & e^{-\overline{z}}e^{\overline{z}}\alpha\otimes\alpha = \alpha\otimes\alpha\,, \\ e^{i\mathrm{Im}z(T_l\otimes 1-1\otimes T_l)}(\alpha\otimes\alpha)e^{-i\mathrm{Im}z(T_l\otimes 1-1\otimes T_l)} & = & (\lambda_{z,0}\otimes\lambda_{-z,0})(\alpha\otimes\alpha)(\lambda_{z,0}\otimes\lambda_{-z,0})^* \\ & = & e^{-z}e^{z}\alpha\otimes\alpha = \alpha\otimes\alpha. \end{array}$$

By definition of $\hat{\Delta}_x$, we have

$$\hat{\Delta}_x(\hat{B}) = \hat{\Delta}_x(e^{x\mathrm{Re}T_l}|\beta|) = (e^{x\mathrm{Re}T_l}\otimes 1)\Upsilon|\alpha\otimes\beta+\beta\otimes 1|\Upsilon^*,$$

$$\hat{\Delta}_x(\hat{v}) = \hat{\Delta}_x(e^{-ix\mathrm{Im}T_l}\mathrm{Ph}(\beta)) = (e^{-ix\mathrm{Im}T_l}\otimes 1)\Upsilon\mathrm{Ph}(\alpha\otimes\beta+\beta\otimes 1)\Upsilon^*.$$

A direct computation gives

$$\mathrm{Ph}(\alpha\otimes\beta+\beta\otimes 1)(e^{x\mathrm{Re}T_l}\otimes 1) = e^x(e^{x\mathrm{Re}T_l}\otimes 1)\mathrm{Ph}(\alpha\otimes\beta+\beta\otimes 1),$$

so

$$\begin{array}{rcl} \hat{\Delta}_x(\hat{\beta}) & = & e^x(e^{xT_l^*}\otimes 1)\Upsilon(\alpha\otimes\beta+\beta\otimes 1)\Upsilon^* \\ & = & e^x(e^{xT_l^*}\otimes 1)\Upsilon(\alpha\otimes\beta)\Upsilon^* \dot{+} e^x(e^{xT_l^*}\otimes 1)\Upsilon(\beta\otimes 1)\Upsilon^*. \end{array}$$

Thus, it suffices to show that

$$\hat{\alpha}\otimes\hat{\beta} = e^x(e^{xT_l^*}\otimes 1)\Upsilon(\alpha\otimes\beta)\Upsilon^* \tag{20}$$

$$\hat{\beta}\otimes 1 = e^x(e^{xT_l^*}\otimes 1)\Upsilon(\beta\otimes 1)\Upsilon^*. \tag{21}$$

Let us prove (20). Let us put $T = e^x e^{xT_l^*}\otimes 1 = e^x L\otimes 1$ and $S = \Upsilon(\alpha\otimes\beta)\Upsilon^*$. We want to show that $\hat{\alpha}\otimes\hat{\beta} = TS$. For all $z\in\mathbb{C}$, we have

$$e^{ix\mathrm{Im}z(T_r\otimes 1)}(\alpha\otimes 1)e^{-ix\mathrm{Im}z(T_r\otimes 1)} = (\lambda_{0,xz}\alpha\lambda_{0,-xz}^*\otimes 1) = e^{xz}(\alpha\otimes 1),$$

and, using the fact that $\alpha\otimes 1$ and $1\otimes T_l^*$ strongly commute, we obtain $\Upsilon(\alpha\otimes 1)\Upsilon^* = \alpha\otimes e^{xT_l^*} = \alpha\otimes L$. Similarly, $\Upsilon(1\otimes\beta)\Upsilon^* = R\otimes\beta$. Thus, using Lemma 14, we see that the polar decomposition of S is

$$\mathrm{Ph}(S) = \mathrm{Ph}(\alpha)\mathrm{Ph}(R)\otimes\mathrm{Ph}(\beta)\mathrm{Ph}(L), \quad |S| = |\alpha||R|\otimes|\beta||L|.$$

Moreover, the polar decomposition of T is given by $\mathrm{Ph}(T) = \mathrm{Ph}(L)\otimes 1$, $\quad |T| = e^x|L|\otimes 1$, so, using Lemma 14, one can see that (T,S) is a e^x-commuting pair. In particular, the polar decomposition of TS is

$$|TS| = e^{-x}|T||S| = |L||\alpha||R|\otimes|\beta||L|,\ \mathrm{Ph}(TS) = \mathrm{Ph}(L)\mathrm{Ph}(\alpha)\mathrm{Ph}(R)\otimes\mathrm{Ph}(\beta)\mathrm{Ph}(L).$$

But Proposition 12 gives $\mathrm{Ph}(\hat{\alpha}) = \mathrm{Ph}(L)\mathrm{Ph}(R)\mathrm{Ph}(\alpha)$ and $|\alpha| = |L||R||\alpha|$. Thus, we conclude that $\mathrm{Ph}(\hat{\alpha} \otimes \hat{\beta}) = \mathrm{Ph}(TS)$ and $|\hat{\alpha} \otimes \hat{\beta}| = |TS|$ which concludes the proof of (20). One can prove (21) similarly. ■

Now the proof of Theorem 3 follows: Proposition 11 says that $\hat{\alpha}$ and $\hat{\beta}$ generate $\widehat{M}_x$ and Proposition 12 gives the commutation relations for $\hat{\alpha}$ and $\hat{\beta}$.

5.2 Twisting of the quantum $az + b$ group

This Section is devoted to the proof of Theorem 4. Let $0 < q < 1$ and (M, Δ) be the $az + b$ Woronowicz' quantum group. Let $\alpha : L^\infty(\mathbb{C}^q) \to M$ be defined by $\alpha(F) = F(a)$. Recall that (Section 2.7) $\widehat{\mathbb{C}^q} < (M, \Delta)$ is an abelian stable co-subgroup with the morphism $\gamma_t = q^{2it} \in \mathbb{C}^q$. Let us perform the twisting construction using the bicharacters

$$\Psi_x(q^{k+i\varphi}, q^{l+i\psi}) = q^{ix(k\psi - l\varphi)}, \ \forall x \in \mathbb{Z},$$

and let (M_x, Δ_x) be the twisted l.c. quantum group.

Proposition 14

$$\Delta_x(a) = a \otimes a \quad \textit{and} \quad \Delta_x(b) = u^{-x+1}|a|^{x+1} \otimes b \dot{+} b \otimes u^x|a|^{-x},$$

and $[D\varphi_x : D\varphi]_t = A_x^{it} = |a|^{-2ixt}$*. The modular element* $\delta_x = |a|^{4x+2}$*, the antipode is not deformed. If* $x, y \in \mathbb{N}$ *and* $x \neq y$*, then* (M_x, Δ_x) *and* (M_y, Δ_y) *are not isomorphic; if* $x \neq 0$*, then* (M_x, Δ_x) *and* (M_{-x}, Δ_{-x}) *are not isomorphic.*

Proof. The relations of commutation from Preliminaries give

$$\begin{aligned}
\Psi_x(a, q^{l+i\psi})b &= \Psi_x(u, q^{l+i\psi})\Psi_x(|a|, q^{l+i\psi})b \\
&= u^{-xl}|a|^{ix\psi}v|b| \\
&= q^{ix\psi - xl}v|b|u^{-xl}|a|^{ix\psi} \\
&= q^{ix\psi - xl}b\Psi_x(a, q^{l+i\psi}).
\end{aligned}$$

So, for any $\gamma \in \mathbb{C}^q$, one has

$$\Psi_x(a \otimes 1, \gamma)(b \otimes 1)\Psi_x(a \otimes 1, \gamma)^* = (\mathrm{Phase}(\gamma))^x \, |\gamma|^{-x}(b \otimes 1).$$

Put $\Omega_x = (\alpha \otimes \alpha)(\Psi_x) = \Psi_x(a \otimes 1, 1 \otimes a)$. Using the previous formula and the fact that $b \otimes 1$ and $1 \otimes a$ strongly commute, one gets $\Omega_x(b \otimes 1)\Omega_x^* = b \otimes u^x|a|^{-x}$. Similarly: $\Omega_x(1 \otimes b)\Omega_x^* = u^{-x}|a|^x \otimes b$. These formulas give the comultiplication on b. The comultiplication on a is clear. We Since $\Psi_s(\gamma_t, \gamma_s) = 1$, for all $s, t \in \mathbb{R}$, then $[D\varphi_x : D\varphi]_t = A_x^{it} = \Psi_x(a, \gamma_t^{-1}) = |a|^{-2ixt}$. Put $f_t^x = \Psi_x(\cdot, \gamma_t)\Psi_x(\gamma_t^{-1}, \cdot)$, then $f_t^x(q^{k+i\varphi}) = q^{4itxk}$ and $\alpha(f_t^x) = |a|^{4itx}$. So, the modular element is $\delta_x = |a|^2|a|^{4x}$. The antipode is not deformed because $\Psi_t(x^{-1}, x) = 1$, for any x. The spectrum of the modular element is $\mathrm{Sp}(\delta_x) = q_x^{\mathbb{Z}} \cup \{0\}$, where $q_x = q^{4x+2}$, so, if $x \neq y$ are strictly positive, then $0 < q_x \neq q_y < 1$, so $q_x^{\mathbb{Z}} \neq q_y^{\mathbb{Z}}$, then

(M_x, Δ_x) and (M_y, Δ_y) are not isomorphic. Moreover, if $x > 0$, then (M_x, Δ_x) is not isomorphic to (M_{-x}, Δ_{-x}) because in the opposite case we would have $q^{(4x+2)\mathbb{Z}} = q^{(4x-2)\mathbb{Z}}$, from where, as $x > 0$, $4x + 2 = 4x - 2$ - contradiction. ■

The group $\mathbb{C}^q$ is selfdual with the duality $(q^{k+i\varphi}, q^{l+i\psi}) \mapsto q^{i(k\psi+l\varphi)}$, so one can compute the representations L and R of $\mathbb{C}^q$:

$$L_{q^{k+i\varphi}} = m^{i\varphi} \otimes s^{-k} \otimes 1 \otimes s^k, \quad R_{q^{k+i\varphi}} = m^{-i\varphi} \otimes 1 \otimes m^{i\varphi} \otimes s^{-k}.$$

Then the left-right action of $(\mathbb{C}^q)^2$ on the generators of $\widehat{M}$ is

$$\alpha_{q^{k+i\varphi},q^{l+i\psi}}(\hat{a}) = q^{l-k+i(\psi-\varphi)}\hat{a}, \quad \alpha_{q^{k+i\varphi},q^{l+i\psi}}(\hat{b}) = q^{-k-i\varphi}\hat{b}. \tag{22}$$

Let $N = (\mathbb{C}^q)^2 \ltimes \hat{M}$, it is generated by the operators $\lambda_{q^{k+i\varphi},q^{l+i\psi}}$ and $\pi(x)$, for $x \in \widehat{M}$, and θ be the dual action of $(\mathbb{C}^q)^2$ on N. The deformed dual action is

$$\theta^{\Psi_x}_{q^{k+i\varphi},q^{l+i\psi}} = \lambda_{q^{x(k-i\varphi)},q^{x(-l+i\psi)}}\theta_{q^{k+i\varphi},q^{l+i\psi}}(.)\lambda^*_{q^{x(k-i\varphi)},q^{x(-l+i\psi)}}.$$

Let $\widehat{M}_x$ be the fixed point algebra. The left-right action is very similar to the one for the classical $az + b$. Define $\alpha = \pi(\hat{a})$, $\beta = \pi(\hat{b})$. Then α and β are normal operators affiliated with N and one can see that

$$\theta^{\Psi_x}_{q^{k+i\varphi},q^{l+i\psi}}(\alpha) = q^{-x(l+k)+ix(\varphi+\psi)}\alpha \ , \quad \theta^{\Psi_x}_{q^{k+i\varphi},q^{l+i\psi}}(\beta) = q^{-xk+ix\varphi}\beta. \tag{23}$$

Let T_l and T_r be the "infinitesimal generators" of the left and right translations, so T_l and T_r are affiliated with N and

$$\lambda_{q^{k+i\varphi},q^{l+i\psi}} = (\mathrm{Ph}T_l)^k\,|T_l|^{i\varphi}\,(\mathrm{Ph}T_r)^l\,|T_r|^{i\psi}. \tag{24}$$

Then $\lambda(f) = f(T_l, T_r)\ \forall f \in L^\infty\left((\mathbb{C}^q)^2\right)$. Let $U = \lambda(\Psi_x)$ and $\hat{\alpha} = U^*\alpha U$.

Proposition 15 *$(T_l^*T_r^*)^{-x}$ and α strongly commute and $\hat{\alpha} = (T_l^*T_r^*)^{-x}\,\alpha$. The polar decomposition of $\hat{\alpha}$ is $\hat{u} := Ph\hat{\alpha} = (PhT_lT_r)^x$, $\hat{A} := |\hat{\alpha}| = |T_lT_r|^{-x}|\alpha|$. Also, $|T_l|$ and $|\beta|$ strongly commute, so we can define a positive operator $\hat{B} = |T_l|^{-x}|\beta|$. Let $\hat{v} = Ph(T_l)^x Ph(\beta)$. Then $\hat{\alpha}$ and $\hat{B}$ are affiliated with $\widehat{M}_x$, $\hat{v} \in \widehat{M}_x$, and we have the following relations of commutation:*

- *$\hat{u}\hat{v} = \hat{v}\hat{u}$, $\hat{A}\hat{B} = \hat{B}\hat{A}$;*
- *$\hat{v}\hat{B}\hat{v}^* = q^{-2x}\hat{B}$, $\hat{u}\hat{B}\hat{u}^* = q^{-2x+1}\hat{B}$ and, $\hat{v}\hat{A}\hat{v}^* = q^{-2x-1}\hat{A}$.*

Moreover, these three operators generate $\widehat{M}_x$ in the sense that

$$\widehat{M}_x = \left\{ f(\hat{\alpha})g(\hat{v})h(\hat{B}),\ f \in L^\infty(\mathbb{C}^q),\ g \in L^\infty(\mathbb{S}^1),\ h \in L^\infty(q^{\mathbb{Z}}) \right\}''.$$

Proof. Using (22) and (24), we find:

$$|T_lT_r|^{is}\alpha|T_lT_r|^{-is} = \alpha, \tag{25}$$
$$\mathrm{Ph}(T_lT_r)\alpha\mathrm{Ph}(T_lT_r)^* = \alpha, \tag{26}$$
$$|T_l|^{is}\beta|T_l|^{-is} = q^{-is}\beta, \tag{27}$$
$$\mathrm{Ph}(T_l)\beta\mathrm{Ph}(T_l)^* = q^{-1}\beta. \tag{28}$$

Due to (25) and (26), α and $T_l^*T_r^*$ strongly commute. Because $\Psi_x(T_r, T_r) = 1$, we have $\hat{\alpha} = \Psi_x(T_lT_r, T_r)^*\alpha\Psi_x(T_lT_r, T_r)$. Next, using $\Psi_x(q^{k+i\varphi}, T_r)^*\alpha\Psi_x(q^{k+i\varphi}, T_r) = \lambda_{1,q^{-xk+ix\varphi}}\alpha\lambda^*_{1,q^{-xk+ix\varphi}} = q^{-xk+ix\varphi}\alpha$, and because T_lT_r and α strongly commute, we have

$$\hat{\alpha} = |T_lT_r|^{-x}\left(\mathrm{Ph}T_lT_r\right)^x\alpha = (T_l^*T_r^*)^{-x}\alpha.$$

The polar decomposition of $\hat{\alpha}$ follows. Equality (27) implies that $|T_l|$ and $|\beta|$ strongly commute. Note that

$$\begin{aligned}\theta^{\Psi_x}_{q^{k+i\varphi},q^{l+i\psi}}(U) &= \Psi_x(T_lq^{-k-i\varphi}, T_rq^{-l-i\psi}) \\ &= U\lambda_{q^{xl-ix\psi},q^{-xk+ix\varphi}}\Psi_x(q^{k+i\varphi}, q^{l+i\psi}).\end{aligned}$$

Then, it follows from (22) and (23) that $\hat{\alpha}$ is affiliated with $\widehat{M}_x$. Also, using (23) we find $\theta^{\Psi_x}_{q^{k+i\varphi},q^{l+i\psi}}(\hat{v}) = \left(\mathrm{Ph}(T_lq^{-k-i\varphi})\right)^x q^{ix\varphi}\mathrm{Ph}\beta = \hat{v}$, so $\hat{v} \in \widehat{M}_x$. In the same way we prove that $\hat{B}$ is affiliated with $\widehat{M}_x$. It is easy to see that $\mathrm{Ph}T_l$ and $\mathrm{Ph}T_r$ commute with $\mathrm{Ph}\alpha$ and $\mathrm{Ph}\beta$, and because $\mathrm{Ph}\alpha$ and $\mathrm{Ph}\beta$ commute, it follows that $\hat{u}\hat{v} = \hat{v}\hat{u}$. Also, $|T_l|$ and $|T_r|$ strongly commute with $|\alpha|$ and $|\beta|$, and because $|\alpha|$ and $|\beta|$ strongly commute, it follows that $\hat{A}\hat{B} = \hat{B}\hat{A}$. The relation $\hat{v}\hat{B}\hat{v}^* = q^{-2x}\hat{B}$ follows from (27) and (28). Remark that

$$\mathrm{Ph}\alpha|T_l|^{-x}\mathrm{Ph}\alpha^* = q^{-x}|T_l|^{-x}, \quad \mathrm{Ph}\beta|T_lT_r|^{-x}\mathrm{Ph}\beta^* = q^{-x}|T_lT_r|^{-x},$$

and the two last relations follow from $\mathrm{Ph}\alpha|\beta|\mathrm{Ph}\alpha^* = q|\beta|$ and $\mathrm{Ph}\beta|\alpha|\mathrm{Ph}\beta^* = q^{-1}|\beta|$. The generating property is proved as in Proposition 11. ■

Let $\hat{\Delta}_x$ be the comultiplication on $\widehat{M}_x$ and $\hat{\beta} = \hat{v}\hat{B}$. Then $\hat{\beta}$ is a closed (non-normal) operator affiliated with $\widehat{M}_x$. As before, we define $\hat{\Delta}_x(\hat{\beta}) = \hat{\Delta}_x(\hat{v})\hat{\Delta}_x(\hat{B})$ which is closed, non-normal and affiliated with $\widehat{M}_x \otimes \widehat{M}_x$. The proof of the following Proposition is similar to the one of Proposition 13.

Proposition 16

$$\hat{\Delta}_x(\hat{\alpha}) = \hat{\alpha}\otimes\hat{\alpha} \quad \textit{and} \quad \hat{\Delta}_x(\hat{\beta}) = \hat{\alpha}\otimes\hat{\beta}\dot{+}\hat{\beta}\otimes 1.$$

The proof of Theorem 4 follows from the results of this section.

6 Appendix

Let α be an action of a l.c. quantum group (M, Δ) on the von Neumann algebra N. Let θ be a n.s.f. weight on N and suppose that N acts on a Hilbert space K such that $(K, \iota, \Lambda_\theta)$ is the G.N.S. construction for θ. We define

$$\mathcal{D}_0 = \text{span}\{(a \otimes 1)\alpha(x) \,|\, a \in \mathcal{N}_{\hat{\varphi}}, x \in \mathcal{N}_\theta\}.$$

Let (H, ι, Λ) be the G.N.S. construction for the left invariant weight φ of (M, Δ), $\hat{\varphi}$ the dual weight, and $\hat{\Lambda}$ its canonical G.N.S.-map. We recall that the G.N.S. construction for the dual weight $\tilde{\theta}$ is given by $(H \otimes K, \iota, \tilde{\Lambda})$, where $\tilde{\Lambda}_\theta$ is the σ-strong*-norm closure of the map

$$\mathcal{D}_0 \to H \otimes K \,:\, (a \otimes 1)\alpha(x) \mapsto \hat{\Lambda}(a) \otimes \Lambda_\theta(x).$$

Proposition 17 *Let C_1 be a σ-strong*-norm core for $\hat{\Lambda}$ and C_2 a σ-strong*-norm core for Λ_θ. Then the set $\mathcal{C} =$ span$\{(a \otimes 1)\alpha(x) \,|\, a \in C_1, x \in C_2\}$ is a σ-weak-weak core for $\tilde{\Lambda}_\theta$.*

Proof. Let $a \in \mathcal{N}_{\hat{\varphi}}$ and $x \in \mathcal{N}_\theta$. There exists two nets (a_i) and (x_i), with $a_i \in C_1$ and $x_i \in C_2$, such that

$$a_i \to a,\; x_i \to x \quad \sigma - \text{strongly} * \quad \text{and } \hat{\Lambda}(a_i) \to \hat{\Lambda}(a),\; \Lambda_\theta(x_i) \to \Lambda_\theta(x).$$

Thus, $(a_i \otimes 1)\alpha(x_i) \to (a \otimes 1)\alpha(x)$ σ-weakly and

$$\tilde{\Lambda}_\theta\left((a_i \otimes 1)\alpha(x_i)\right) = \hat{\Lambda}(a_i) \otimes \Lambda_\theta(x_i) \to \hat{\Lambda}(a) \otimes \Lambda_\theta(x) = \tilde{\Lambda}_\theta\left((a \otimes 1)\alpha(x)\right).$$

■

Proposition 18 *Let M be a von Neumann algebra with a n.s.f. weight φ, (H, ι, Λ) the G.N.S. construction for φ, and T a positive self-adjoint operator affiliated with M. Then $\mathcal{C} = \{x \in \mathcal{N}_\varphi \,|\, Tx \;\text{ is bounded and }\; \Lambda(x) \in \mathcal{D}(T)\}$ is a σ-strong*-norm core for Λ and, if $x \in \mathcal{C}$, then $\overline{Tx} \in \mathcal{N}_\varphi$ and $\Lambda(\overline{Tx}) = T\Lambda(x)$.*

Proof. Let $T = \int_0^{+\infty} \lambda de_\lambda$ be the spectral decomposition of T. Let $e_n = \int_0^n de_\lambda$. Then $e_n \to 1$ σ-strongly*, Te_n is bounded with domain H. Let $x \in \mathcal{N}_\varphi$ and put $x_n = e_n x$. We have $x_n \to x$ σ-strongly* and $\Lambda(x_n) = e_n\Lambda(x) \to \Lambda(x)$ in norm. Moreover, $Tx_n = Te_n x$ is bounded and $\Lambda(x_n) = e_n\Lambda(x) \in \mathcal{D}(T)$, so $x_n \in \mathcal{C}$, and it follows that $\mathcal{C}$ is a σ-strong*-norm core for Λ. Now let $x \in \mathcal{C}$. Note that $e_n\overline{Tx} = Te_n x = \overline{e_n T}x$ is in $\mathcal{N}_\varphi$ and it converges σ-strongly* to $\overline{Tx}$. Moreover,

$$\Lambda(e_n\overline{Tx}) = \overline{e_n T}\Lambda(x) = e_n T\Lambda(x) \to T\Lambda(x).$$

Because Λ is σ-strong*-norm closed, we have $\overline{Tx} \in \mathcal{N}_\varphi$ and $\Lambda(\overline{Tx}) = T\Lambda(x)$. ■

Proposition 19 *Let M be a von Neumann algebra, φ_1 and φ_2 two n.s.f. weights on M having the same modular group. Let (H_i, π_i, Λ_i) be the G.N.S. construction for φ_i ($i = 1, 2$). Suppose that there exist a σ-weak-weak core $\mathcal{C}$ for Λ_1 such that $\mathcal{C} \subset \mathcal{N}_{\varphi_1} \cap \mathcal{N}_{\varphi_2}$ and a unitary $Z : H_1 \to H_2$ such that $\Lambda_2(x) = Z\Lambda_1(x)$, for all $x \in \mathcal{C}$. Then $\varphi_1 = \varphi_2$.*

Proof. Because $\mathcal{C}$ is a σ-weak-weak core for Λ_1 and because Λ_2 is σ-weak-weak closed, we have $\mathcal{N}_{\varphi_1} \subset \mathcal{N}_{\varphi_2}$ and, for all $x \in \mathcal{N}_{\varphi_1}$ we have $\Lambda_1(x) = Z\Lambda_2(x)$. Thus, $\varphi_1(y^*x) = \varphi_2(y^*x)$, for all $x, y \in \mathcal{N}_{\varphi_1}$. Let $\mathcal{B} = \mathcal{N}_{\varphi_1}^* \mathcal{N}_{\varphi_1}$. This is a dense *-subalgebra of $\mathcal{M}_{\varphi_1} \cap \mathcal{M}_{\varphi_2}$ and, for all $x \in \mathcal{B}$, we have $\varphi_1(x) = \varphi_2(x)$. Because φ_1 and φ_2 have the same modular group, we can use the Pedersen-Takesaki Theorem [13] to conclude the proof. ■

Let M be a von Neumann algebra, φ a n.s.f. weight on M, (H, ι, Λ) the G.N.S. construction for φ, and σ the modular group of φ. Let δ be a positive self-adjoint operator affiliated with M, $\lambda > 0$ such that $\sigma_t(\delta^{is}) = \lambda^{ist}\delta^{is}$, and Λ_δ the canonical G.N.S. map of the Vaes' weight φ_δ. One can consider on $M \otimes M$ two n.s.f. weights: $\varphi_\delta \otimes \varphi_\delta$, with the canonical G.N.S. map $\Lambda_\delta \otimes \Lambda_\delta$, and the Vaes' weight $(\varphi \otimes \varphi)_{\delta\otimes\delta}$ associated with $\varphi \otimes \varphi$, $\delta \otimes \delta$ and λ^2. Let $\Lambda \otimes \Lambda$ be the G.N.S. map for $\varphi \otimes \varphi$, and $(\Lambda \otimes \Lambda)_{\delta\otimes\delta}$ the G.N.S. map for $(\varphi \otimes \varphi)_{\delta\otimes\delta}$ (see Section 2.6).

Proposition 20 *$\varphi_\delta \otimes \varphi_\delta = (\varphi \otimes \varphi)_{\delta\otimes\delta}$ and $\Lambda_\delta \otimes \Lambda_\delta = (\Lambda \otimes \Lambda)_{\delta\otimes\delta}$.*

Proof. Let us apply the Pedersen-Takesaki theorem to the weights $\varphi_1 := (\varphi_\delta \otimes \varphi_\delta)$ and $\varphi_2 := (\varphi \otimes \varphi)_{\delta\otimes\delta}$ which have the same modular group and are equal on the dense *-subalgebra $\mathcal{B} = N \odot N$ of $\mathcal{M}_{\varphi_1} \cap \mathcal{M}_{\varphi_2}$, where

$$N := \left\{ x \in M \mid x\delta^{\frac{1}{2}} \text{ is bounded and } \overline{x\delta^{\frac{1}{2}}} \in \mathcal{N}_\varphi \right\}.$$

Let Λ_i be the canonical G.N.S. map of φ_i. By definition, $N \odot N$ is a σ-strong*-norm core for Λ_1, and $\Lambda_1|_N = \Lambda_2|_N$. Since Λ_1 and Λ_2 are σ-strongly*-norm closed, then $\Lambda_1 \subset \Lambda_2$. And $\Lambda_1 = \Lambda_2$ since $\mathcal{D}(\Lambda_1) = \mathcal{N}_{\varphi_1} = \mathcal{N}_{\varphi_2} = \mathcal{D}(\Lambda_2)$. ■

Finally, let us formulate the von Neumann algebraic version of [6], Lemma 3.6. Let N be a von Neumann algebra, G a l.c. abelian group, $u : G \to N$ a unitary representation of G and $\theta : \hat{G} \to \mathrm{Aut}(N)$ an action of $\hat{G}$ on N such that

$$\theta_\gamma(u(g)) = \overline{< \gamma, g >} u(g).$$

Let α be the action of G on N implemented by u. The unitary representation u of G gives a *-homomorphism $\pi : L^\infty(\hat{G}) \to N$.

Lemma 15 *Let V be a linear subspace of N^θ invariant under the action α and such that $\left(\pi(L^\infty(\hat{G}))V\pi(L^\infty(\hat{G}))\right)'' = N$. Then $V'' = N^\theta$.*

References

[1] J. Bichon, J., A. De Rijdt, and S. Vaes, Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups, *Commun. Math. Phys.*, 22, 703-728, 2006

[2] M. Enock and J.-M. Schwartz, Kac algebras and duality of locally compact groups, Springer, Berlin, 1992.

[3] M. Enock and L. Vainerman, Deformation of a Kac algebra by an abelian subgroup, *Commun. Math. Phys.*, 178, No. 3, 571-596, 1996

[4] P. Fima, Constructions et exemples de groupes quantiques localement compacts, *Ph.D. Thesis*, Caen, 2007

[5] P. Fima and L. Vainerman, A locally compact quantum group of upper triangular matrices, *to appear in Ukrainian Mathematical Journal*

[6] P. Kasprzak, Rieffel deformation via crossed products, *Preprint: arXiv:math/0606333v11*

[8] J. Kustermans and S. Vaes, Locally compact quantum groups, *Ann. Sci. Ec. Norm. Super.*, IV, Ser. 33, No. 6, 547-934, 2000

[9] J. Kustermans and S. Vaes, Locally compact quantum groups in the von Neumann algebraic setting, *Math. Scand.*, 92 (1), 68-92, 2003

[10] M. Landstad, Quantization arising from abelian subgroups, *Int. J. Math.*, 5, 897-936, 1994

[11] M. Rieffel, Deformation quantization for actions of $\mathbb{R}^d$, *Memoirs A.M.S.*, 506, 1993

[12] M. Rieffel, Non-compact quantum groups associated with abelian subgroups, *Comm. Math. Phys.*, 171 (1), 181-201, 1995

[13] M. Takesaki, Theory of Operator Algebras II, *Springer*, 2002

[14] S. Vaes, A Radon-Nikodym theorem for von Neuman algebras, *J. Operator. Theory.*, 46, No.3, 477-489, 2001

[15] S. Vaes, Locally compact quantum groups, *Ph.D. Thesis*, Leuven, 2001

[16] L. Vainerman, 2-cocycles and twisting of Kac algebras, *Commun. Math. Phys.*, 191, No. 3, 697-721, 1998

[17] A. Van Daele, The Haar measure on some locally compact quantum groups, *preprint : arXiv:math.OA/0109004*

[18] S. L. Woronowicz, Operator Equalities Related to the Quantum $E(2)$ Group, *Commun. Math. Phys.*, 144, 417-428, 1992

[19] S. L. Woronowicz, Quantum $az + b$ group on complex plane, *Int. J. Math.*, 12, No. 4, 461-503, 2001